Rockafellar and Wets: Variational Analysis 4.G

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命題 (4.30 (a) [RW]): *If* $K^\nu \to K$ *for cones* $K^\nu, K \subseteq \mathbb{R}^n$ *such that* K *is pointed, then* $\operatorname{con} K^\nu \to \operatorname{con} K$.

(証明) 任意の cone C に対して

$$\operatorname{con} C = C + \cdots + C \ (n \text{ 項})$$

である (3.15) ので, 4.29(d) を適用することを考えよう.

- cone の場合は完全収束と普通の収束は同値 (4.25) だから, $K^\nu \xrightarrow{\mathrm{t}} K$.
- K は閉なので, 3.15 より $\operatorname{con} K$ は閉.
- K は closed cone なので, $K \times \cdots \times K$ も closed cone. よって,

$$K^\infty \times \cdots \times K^\infty = K \times \cdots \times K = (K \times \cdots \times K)^\infty.$$

- $x_1 + \cdots + x_n = \mathbf{0},\ x_i \in K^\infty = K\ (i = 1, \cdots, n)$

$$\Longrightarrow x_1 = \cdots = x_n = \mathbf{0} \quad \text{(by the pointedness of } K\text{)}.$$

ゆえに,

$$\operatorname{con} K^\nu = \overbrace{K^\nu + \cdots + K^\nu}^{n\text{ 個}} \xrightarrow{\mathrm{t}} \overbrace{K + \cdots + K}^{n\text{ 個}} = \operatorname{con} K. \ \square$$

The following is a slight generalization of Proposition 3.14.

補題: *A cone* $K \subseteq \mathbb{R}^n$ *is pointed if and only if* $\operatorname{con} K$ *is pointed, i.e., the subspace* $(\operatorname{con} K) \cap (-\operatorname{con} K)$ *contains a nonzero vector.*

(証明) First, note that $(\operatorname{con} K) \cap (-\operatorname{con} K)$ is indeed a subspace. Suppose that $\mathbf{0} \neq x \in (\operatorname{con} K) \cap (-\operatorname{con} K)$. Then, since $x \in \operatorname{con} K$,

$$x = \lambda_1 y_1 + \cdots + \lambda_k y_k$$

for some $\lambda_i > 0$ $(i = 1, \cdots, k)$ and some $\mathbf{0} \neq x_i \in K$ $(i = 1, \cdots, k)$. Also, since $-x \in \operatorname{con} K$ we have

$$-x = \lambda_{k+1} y_{k+1} + \cdots + \lambda_l y_l$$

for some $\lambda_i > 0$ $(i = k+1, \cdots, l)$ and some $\mathbf{0} \neq x_i \in K$ $(i = k+1, \cdots, l)$. Since K is a cone, we have $x_i := \lambda_i y_i \in K$ $(i = 1, \cdots, l)$ and have a expression

$$\mathbf{0} = x_1 + \cdots + x_l, \tag{1}$$

where $x_i \neq \mathbf{0}$ for each $i = 1, \cdots, l$. Hence, K is not pointed.

Conversely, suppose that K is not pointed. Then, we have (1) for $x_i \neq \mathbf{0}$ $(i = 1, \cdots, l)$. Then, we must have $l \geq 1$ and the expression

$$-x_1 = x_2 + \cdots + x_l.$$

the left-hand side is an element of $-K \subseteq -\operatorname{con} K$, whereas the right-hand side is that of $\operatorname{con} K$. It follows that $\mathbf{0} \neq -x_1 \in (\operatorname{con} K) \cap (-\operatorname{con} K)$. $\square$

補題: *For a cone $K \subseteq \mathbb{R}^n$ K is pointed if there exists a nonzero vector a such that*

$$\{x \mid ax \leq 0\} \cap K = \{\mathbf{0}\}. \tag{2}$$

(証明) Suppose we have

$$x_1 + \cdots + x_k = \mathbf{0}, x_i \in K \ (i = 1, \cdots, k).$$

すべての $i = 1, \cdots, k$ に対して $ax_i \geq 0$ である．なぜなら，もし，$ax_i < 0$ なる i があれば、(2) によって $x_i = \mathbf{0}$ であるので，$ax_i = 0$ これは矛盾。

一方で，

$$ax_1 + \cdots + ax_k = 0$$

なので，$ax_i = 0 \ (i = 1, \cdots, k)$。再び、(2) によって $x_i = \mathbf{0} \ (i = 1, \cdots, k)$. 即ち、$K$ は pointed. □

Note that the converse direction of the above lemma does not hold: Consider the "pointed" cone $\{(0, \xi_2) \mid \xi_2 \leq 0\} \cup \{(\xi_1, \xi_2) \mid \xi_1 > 0\}$.

命題: *For an arbitrary subset $S \subseteq \mathbb{R}^n$, we have*

$$\text{con cl}\, S \subseteq \text{cl con}\, S \quad \textit{and} \quad \text{cl con cl}\, S = \text{cl con}\, S$$

(証明) If $x \in \text{con cl}\, S$, then

$$x = \lambda_1 x_1 + \cdots + \lambda_k x_k$$

for some $\lambda_1, \cdots, \lambda_k \geq 0$ with $\sum_{i=1}^k = 1$ and $x_1, \cdots, x_k \in \text{cl}\, S$. Then, for each $i = 1, \cdots, k$ there exists sequence $(x_i^\nu \mid \nu = 1, 2, \cdots)$ in S converging to x_i. Let $x^\nu = \sum_{i=1}^k \lambda_i x_i^\nu$. Then $x^\nu \in \text{con}\, S$ and $x^\nu \longrightarrow x$. Hence, $x \in \text{cl con}\, S$.

Since $\text{con cl}\, C \subseteq \text{cl con}\, C$, we have $\text{cl con cl}\, C \subseteq \text{cl con}\, C$. The converse inclusion is trivial. □

Define $K^* \subseteq \mathbb{R}^n$ by

$$K^* = \{y \mid \forall x \in K \colon \langle y, x \rangle \leq 0\},$$

where $\langle \cdot, \cdot \rangle$ denotes canonical inner product.

命題: *(i) $K^* = (\text{con}\, K)^*$.*
(ii) For a closed cone K, if K is pointed then $K^{**} = \text{con}\, K$.

(証明) Since $K \subseteq \text{con}\, K$, $K^* \supseteq (\text{con}\, K)^*$ is obvious. Suppose $x \in K^*$ and let $y \in \text{con}\, K$ be arbitrary. Since there exists $\lambda_1, \cdots, \lambda_k \geq 0$ with $\sum_{i=1}^k \lambda_i = 1$ and $z_i \in K \ (i = 1, \cdots, k)$ such that $y = \sum_{i=1}^k \lambda_i z_i$, we have

$$\langle x, y \rangle = \langle x, \sum_{i=1}^k \lambda_i z_i \rangle = \sum_{i=1}^k \lambda_i \langle x, z_i \rangle \leq 0,$$

and hence, $x \in (\text{con}\, K)^*$. This shows $K^* \subseteq (\text{con}\, K)^*$.

If a closed cone K is pointed, then $\text{con}\, K$ is again closed (3.15) (and pointed). Hence, $K^{**} = (\text{con}\, K)^{**}$. □

補題 (Theorem 2.7.7 [SW]): *For a closed convex cone $K \subseteq \mathbb{R}^n$ we have $K^{**} = K$.*

(証明) It suffices to show $K^{**} \subseteq K$. Suppose there exists $x \in K^{**} \backslash K$. Then, by Theorem 2.39 (Strong Separation) there exists a halfspace $H^- = \{x \mid \langle a, x \rangle \leq \beta\}$ such that

$$K \subseteq H^- \ \text{ but } \ \langle a, x \rangle > \beta. \tag{3}$$

Since $\mathbf{0} \in K$, we must have $0 \leq \beta$.

We will show that we can take $\beta = 0$. Suppose that there exists $y \in K$ such that $\langle a, y \rangle > 0$. Then, for sufficiently large $\lambda > 0$, we must have $\langle z, \lambda y \rangle > \beta$. However, since $\lambda y \in K$, this contradicts $K \subseteq H^-$.

We have $a \in K^*$ and, since $x \in K^{**}$, we must have $\langle a, x \rangle \leq 0$. This is a contradiction. □

Note that the above lemma, when specialized to polyhedral (convex) cones, yields Farkas' lemma.

命題: *For a closed convex cone $K \subseteq \mathbb{R}^n$, K is pointed if and only if K^* is full-dimensional.*

(証明) (“only if “ part:) Suppose that K^* is not full-dimensional then there exists a homogeneous hyperplane $H = \{x \mid \langle a, x\rangle = 0\}$ such that $K^* \subseteq H$. Then, $K^{**} = K$ contains one dimensional subspace $H^* = \{\lambda a \mid \lambda \in \mathbb{R}\}$. So does $K \cap (-K)$.

(“if” part:) Conversely, if K is not pointed then $K \cap (-K) \subseteq K$ contains one-dimensional subspace H^*. Then $K^* \subseteq H^{**} = H$, and hence, K^* is not full-dimensional. □

補題: *For a closed cone $K \subseteq \mathbb{R}^n$ K is pointed if and only if there exists a nonzero vector a such that*

$$\{x \mid ax \leq 0\} \cap K = \{\mathbf{0}\}. \tag{4}$$

(証明) It suffices to show the “only if” part. Suppose that K is pointed. Then, $\mathrm{con}\, K$ is pointed, and hence, $(\mathrm{con}\, K)^* = K^*$ is full-dimensional.

Let $c_1, \cdots, c_n$ be linearly independent vectors in K^* and let

$$a = c_1 + \cdots + c_n.$$

Define a polyhedral cone $C \subseteq K^*$ by

$$C = \{\lambda_1 c_1 + \cdots + \lambda_n c_n \mid \lambda_1, \cdots, \lambda_n \geq 0\}.$$

Then, $C^* \supseteq K^{**} \supseteq K$ and $C^* = \{x \mid c_i x \leq 0\ (i = 1, \cdots, n)\}$ is pointed. We have $\langle a, x\rangle \leq 0$ for $\forall x \in K$.

Now, suppose $x \in \{x \mid \langle a, x\rangle \geq 0\} \cap K$. Then we have $\langle a, x\rangle = 0$, and hence, $c_i x = 0\ (i = 1, \cdots, n)$. But since $c_1, \cdots, c_n$ are linearly independent, we must have $x = \mathbf{0}$. □

補題: *If $K^\nu \to K$ for cones $K^\nu, K \subseteq \mathbb{R}^n$ such that K is pointed, then all but finite K^ν are pointed.*

(証明) Since K is pointed, there exists $a \neq \mathbf{0}$ such that

$$K \cap \{x \mid ax \geq 0\} = \{\mathbf{0}\}$$

by the lemma above. We will show that for all but finite ν we have

$$K^\nu \cap \{x \mid ax \geq 0\} = \{\mathbf{0}\}$$

from which the present lemma follows. Suppose on the contrary that there exists $N \in \mathcal{N}_\infty^\sharp$ such that

$$K^\nu \cap \{x \mid ax \geq 0\} \neq \{\mathbf{0}\} \quad (\nu \in N).$$

Then, there exists a sequence $\{x^\nu\}_{\nu \in N}$ such that $x^\nu \in K^\nu\ (\nu \in N)$. Since K^ν is a cone for each ν, after appropriate scalings if necessary, we can assume $\|x^\nu\| = 1\ (\nu \in N)$ so that there exists a cluster point $\bar{x}$ with $a\bar{x} \geq 0$. Furthermore, we have $\bar{x} \in \limsup_\nu K^\nu = K$, which is impossible. □

命題 (4.30 (c) [RW]): *If $C^\nu \to C$ for sets $C^\nu, C \subseteq \mathbb{R}^n$ with $C \neq \emptyset$ and C^∞ being pointed, then* $\mathrm{con}\, C^\nu \xrightarrow{\mathrm{t}} \mathrm{cl}\, \mathrm{con}\, C$.

(証明) 任意の $D \subseteq \mathbb{R}^n$ に対して $\tilde{D} \subseteq \mathbb{R}^{n+1}$ を

$$\tilde{D} := \left\{\lambda \begin{bmatrix} x \\ -1 \end{bmatrix} \mid x \in \mathrm{cl}\, D, \lambda > 0\right\} \cup \left\{\begin{bmatrix} x \\ 0 \end{bmatrix} \mid x \in D^\infty\right\}$$

と定義する. これは、射線空間モデルによる $\mathrm{csm}\, D$ $(= \mathrm{cl}\, D \cup \mathrm{dir}\, D^\infty)$ の表現である.

主張: $\tilde{D}$ は pointed であるための必要十分条件は, D^∞ が pointed であることである.

(証明) D^∞ が pointed であるとして, $x_1 + \cdots + x_l = \mathbf{0}$ かつ $x_i \in \tilde{D}$ $(i = 1, \cdots, l)$ とする. すると, すべての i について $x_i \in \left\{\begin{bmatrix} x \\ 0 \end{bmatrix} \mid x \in D^\infty\right\}$ でなければならないので, $x_i = \begin{bmatrix} y_i \\ 0 \end{bmatrix}$ for some $y_i \in D^\infty$. D^∞ は pointed であるから, $y_1 = \cdots = y_l = \mathbf{0}$. したがって, $x_1 = \cdots = x_l = \mathbf{0}$.

逆に $\tilde{D}$ が pointed であるとして, $y_1 + \cdots + y_l = \mathbf{0}$ かつ $y_i \in D^\infty$ $(i = 1, \cdots, l)$ とする. $x_i = \begin{bmatrix} y_i \\ 0 \end{bmatrix}$ $(i = 1, \cdots, l)$ とおくと, $x_1 + \cdots + x_l = \mathbf{0}$ かつ $x_i \in \tilde{D}$ $(i = 1, \cdots, l)$ であるが, $\tilde{D}$ の pointedness から $x_1 = \cdots = x_l = \mathbf{0}$. ゆえに, $y_1 = \cdots = y_n = \mathbf{0}$. □□

$$C^\nu \xrightarrow{\text{t}} C$$

より

$$\text{csm}\, C^\nu \xrightarrow{\text{c}} \text{csm}\, C \quad (i.e., \text{cl}\, C^\nu \cup \text{dir}\, (C^\nu)^\infty \xrightarrow{\text{c}} C \cup \text{dir}\, C^\infty).$$

これは次と同値.

$$\widetilde{C^\nu} \longrightarrow \tilde{C}.$$

ここで, 有限個を除く全ての ν に対して, $\widetilde{C^\nu}$ は pointed したがってそれらの ν に対して, $(C^\nu)^\infty$ は pointed.

$\tilde{C}$ は pointed なので 4.30(a) が使えて,

$$\text{con}\, \widetilde{C^\nu} \longrightarrow \text{con}\, \tilde{C}.$$

これは cosmic space で

$$(\text{con cl}\, C^\nu + \text{con}\, ((\text{cl}\, (C^\nu))^\infty)) \cup \text{dir con}\, ((\text{cl}\, (C^\nu))^\infty) \xrightarrow{\text{c}} (\text{con}\, C + \text{con}\, (C^\infty)) \cup \text{dir con}\, (C^\infty)$$

を意味する。p.81, l.4 より

$$C^\infty = (\text{cl}\, C)^\infty.$$

なので,

$$(\text{con cl}\, C^\nu + \text{con}\, ((C^\nu)^\infty)) \cup \text{dir con}\, ((C^\nu)^\infty) \xrightarrow{\text{c}} (\text{con}\, C + \text{con}\, (C^\infty)) \cup \text{dir con}\, (C^\infty).$$

$\emptyset \neq C = \liminf_\nu C^\nu$ であるから, 有限個を除くすべての ν に対して $C^\nu \neq \emptyset$ でもある. また C^∞ も $(C^\nu)^\infty$ も pointed であるから 3.46 によって

$$\text{con}\, C + \text{con}\, (C^\infty) = \text{cl con}\, C$$

かつ

$$\text{con cl}\, [C^\nu] + \text{con}\, ([C^\nu]^\infty) = \text{cl con}\, (\text{cl}\, [C^\nu]) = \text{cl con}\, [C^\nu].$$

また, 3.46 によって,

$$\text{con}\, ([C^\nu]^\infty) = (\text{cl con}\, [C^\nu])^\infty = (\text{con}\, [C^\nu])^\infty$$

かつ

$$\text{con}\, (C^\infty) = (\text{cl con}\, C)^\infty = (\text{con}\, C)^\infty.$$

ゆえに

$$(\text{cl con}\, C^\nu) \cup \text{dir}\, (\text{con}\, (C^\nu))^\infty \xrightarrow{\text{c}} (\text{cl con}\, C) \cup \text{dir}\, (\text{con}\, C)^\infty.$$

これは, $\text{con}\, C^\nu \xrightarrow{\text{t}} \text{con}\, C$ を意味する。□

命題 (4.30(b)): *If* $C^\nu \longrightarrow C$ *for sets* C^ν *contained in some bounded region of* $\mathbb{R}^n$, *then* $\text{con}\, C^\nu \longrightarrow \text{con}\, C$.

(証明) これは, 4.30(c) で $C^\infty = \{\mathbf{0}\}$ の場合. ($\text{con}\, C$ は閉であることに注意.) □

練習 (4.31): *For* $C_i^\nu \subseteq \mathbb{R}^n$ $(i = 1, \cdots, m)$

$$C_i^\nu \longrightarrow C_i \ (i = 1, \cdots, m) \quad \Longrightarrow \quad \bigcup_{i=1}^m C_i^\nu \longrightarrow \bigcup_{i=1}^m C_i$$

$$C_i^\nu \xrightarrow{\text{t}} C_i \ (i = 1, \cdots, m) \quad \Longrightarrow \quad \bigcup_{i=1}^m C_i^\nu \xrightarrow{\text{t}} \bigcup_{i=1}^m C_i$$

(証明) $\limsup \bigcup_{i=1}^m C_i^\nu \subseteq \bigcup_{i=1}^m C_i \subseteq \liminf \bigcup_{i=1}^m C_i^\nu$ を言う。$x \in \limsup \bigcup_{i=1}^m C_i^\nu$ とする.

$$\exists N \in \mathcal{N}_\infty^\sharp, \exists x^\nu \in \bigcup_{i=1}^m C_i^\nu \ (\nu \in N) \colon x^\nu \underset{N}{\longrightarrow} x.$$

ある $i = 1, \cdots, m$ に対して

$$\exists N \in \mathcal{N}_\infty^\sharp, \exists x^\nu \in C_i^\nu \ (\nu \in N) \colon x^\nu \underset{N}{\longrightarrow} x.$$

即ち,

$$x \in \bigcup_{i=1}^m \limsup_\nu C_i^\nu = \textstyle\bigcup_{i=1}^m C_i = \bigcup_{i=1}^m \liminf_\nu C_i^\nu.$$

$x \in \bigcup_{i=1}^m \liminf_\nu C_i^\nu$ なので,

$$\exists i \in m, \exists N \in \mathcal{N}_\infty, \exists x^\nu \in C_i^\nu \ (\nu \in N) \colon x^\nu \underset{N}{\longrightarrow} x.$$

$$\exists N \in \mathcal{N}_\infty, \exists x^\nu \in \bigcup_{i=1}^m C_i^\nu \ (\nu \in N) \colon x^\nu \underset{N}{\longrightarrow} x.$$

$$x \in \liminf_\nu \textstyle\bigcup_{i=1}^m C_i^\nu.$$

3.9 によって,

$$\limsup{}^\infty \textstyle\bigcup_{i=1}^m C_i^\nu \subseteq \bigcup_{i=1}^m \limsup{}^\infty C_i^\nu \subseteq \bigcup_{i=1}^m C_i^\infty \subseteq \left(\bigcup_{i=1}^m C_i\right)^\infty. \ \square$$

命題 (4(8)): *For a continuous mapping $F \colon \mathbb{R}^n \to \mathbb{R}^m$ and $D^\nu \subseteq \mathbb{R}^m$ we have*

$$\liminf_\nu F^{-1}(D^\nu) \subseteq F^{-1}(\liminf_\nu D^\nu),$$

$$\limsup_\nu F^{-1}(D^\nu) \subseteq F^{-1}(\limsup_\nu D^\nu),$$

(証明) Suppose $x \in \liminf_\nu F^{-1}(D^\nu)$. Then,

$$\exists N \in \mathcal{N}_\infty, \exists x^\nu \in F^{-1}(D^\nu) \ (\nu \in N) \colon x^\nu \underset{N}{\longrightarrow} x.$$

Then, since F is continuous, we have

$$F(x^\nu) \in D^\nu \ (\nu \in N) \text{ and } F(x^\nu) \underset{N}{\longrightarrow} F(x).$$

Hence, $F(x) \in \liminf_\nu D^\nu$, i.e, $x \in F^{-1}(\liminf_\nu D^\nu)$. Similarly for lim sup. $\square$

命題 (4.(9)): *If $\limsup_\nu X^\nu \subseteq X$, $\limsup_\nu D^\nu \subseteq D$ and if for each sequence (x^ν) converging to x we have $F^\nu(x^\nu) \longrightarrow F(x)$, then we have*

$$\limsup_\nu (X^\nu \cap F^{\nu-1}(D^\nu)) \subseteq X \cap F^{-1}(D)$$

(証明) Suppose that $x \in \limsup_\nu (X^\nu \cap F^{\nu-1}(D^\nu))$. Then,

$$\exists N \in \mathcal{N}_\infty^\sharp, \exists x^\nu \in X^\nu \cap F^{\nu-1}(D^\nu) \ (\nu \in N) \colon x^\nu \underset{N}{\longrightarrow} x.$$

Then, $x \in \limsup_\nu X^\nu \subseteq X$. Also, since $F^\nu(x^\nu) \in D^\nu$ $(\nu \in N)$ and $F^\nu(x^\nu) \underset{N}{\longrightarrow} F(x)$, we thus have $F(x) \in \limsup_\nu D^\nu \subseteq D$. Hence, $x \in F^{-1}(D)$. $\square$

命題: *For a linear mappings $L^\nu, L \colon \mathbb{R}^n \to \mathbb{R}^m$, point-wise convergence $L^\nu \longrightarrow L$ is equivalent to $L^\nu(x^\nu) \longrightarrow L(x)$ whenever $x^\nu \longrightarrow x$.*

(証明) Suppose that L^ν and L are represented by matrices $A^\nu, A \in \mathbb{R}^{m\times n}$. Then, the convergence $L^\nu \longrightarrow L$ is equivalent to the convergence of the associated matrices: $A^\nu \longrightarrow A$. [(Proof) Suppose $L^\nu \longrightarrow L$, i.e., for each $x \in \mathbb{R}^n$ we have $A^\nu x \longrightarrow Ax$. Consider the unit vectors e_j $(j = 1, \cdots, n)$: $A^\nu e_j \longrightarrow Ae_j$. This means each column of A^ν converges to the corresponding column. Therefore, $A^\nu \longrightarrow A$. Conversely, suppose $A^\nu \longrightarrow A$. Then for each $x = \xi_1 e_1 + \cdots + \xi_n e_n$ we have

$$A^\nu x = \xi_1 A^\nu e_1 + \cdots + \xi_n A^\nu e_n \longrightarrow \xi_1 Ae_1 + \cdots + \xi_n Ae_n = Ax.]$$

Suppose $L^\nu \longrightarrow L$ and let us consider a sequence converging to x.

$$\|A^\nu x^\nu - Ax\| \le \|A^\nu x^\nu - Ax^\nu\| + \|A(x^\nu - x)\| \longrightarrow 0 \quad (\nu \longrightarrow \infty).$$

Conversely, suppose that for each $x^\nu \longrightarrow x$ we have $A^\nu x^\nu \longrightarrow Ax$. Then, just considering constant sequences (e_j) $(j = 1, \cdots, n)$ we have $A^\nu \longrightarrow A$. □

定理 (4.32): *Let*

$$C^\nu = X^\nu \cap (L^\nu)^{-1}(D^\nu), \quad C = X \cap L^{-1}(D)$$

for linear mapping $L^\nu, L: \mathbb{R}^n \to \mathbb{R}^m$*, convex* $X^\nu, X \subseteq \mathbb{R}^n$ *and convex* $D^\nu, D \subseteq \mathbb{R}^m$ *such that* $L(X)$ *cannot be separated from* D*. If* $L^\nu \longrightarrow L$*,* $\liminf_\nu X^\nu \supseteq X$ *and* $\liminf_\nu D^\nu \supseteq D$*, then* $\liminf_\nu C^\nu \supseteq C$*. Indeed,*

$$L^\nu \longrightarrow L, X^\nu \longrightarrow X, \text{ and } D^\nu \longrightarrow D \Longrightarrow C^\nu \longrightarrow C.$$

(証明) 定理の前半が示されれば後半は自明である.

Suppose that $L^\nu \longrightarrow L$, $\liminf_\nu X^\nu \supseteq X$ and $\liminf_\nu D^\nu \supseteq D$. Let

$$\bar{x} \in C = X \cap L^{-1}(D).$$

僕らは、$x^\nu \in C^\nu$ で $x^\nu \longrightarrow x$ となるものを作らなければいけない。$\bar{x} \in X$ で, $\bar{u} := L(\bar{x}) \in D$ であるので, 点列 $\exists \bar{x}^\nu \in X^\nu$ で $\bar{x}^\nu \longrightarrow \bar{x}$ となる. また $\exists \bar{u}^\nu \in D^\nu$ で $\bar{u}^\nu \longrightarrow \bar{u}$ となる. $\bar{z}^\nu := L(\bar{x}^\nu) - \bar{u}^\nu$ とすると, $\bar{z}^\nu \longrightarrow \mathbf{0}$.

非分離の仮定は 2.39 より, $\mathbf{0} \in \operatorname{int}(L(X) - D)$ と同値. したがって、$\mathbf{0}$ の単体近傍 $S \subseteq \operatorname{int}(L(X) - D)$ が存在する.

$$S = \operatorname{con}(\{z_0, z_1, \cdots, z_m\}),\ z_i = L(x_i) - u_i,\ x_i \in X,\ u_i \in D.$$

$X = \liminf_\nu X^\nu$ かつ $D = \liminf_\nu D^\nu$ であるから,

$$\exists x_i^\nu \in X^\nu : x_i^\nu \longrightarrow x_i \quad (i = 0, 1 \cdots, m) \quad \text{かつ} \quad \exists u_i^\nu \in D^\nu : u_i^\nu \longrightarrow u_i \quad (i = 0, 1, \cdots, m).$$

$z_i^\nu = L(x_i^\nu) - u_i^\nu$ と定義すると, $z_i^\nu \longrightarrow L(x_i) - u_i = z_i$ $(i = 0, 1, \cdots, m)$ であるから, 2.28(f) によって、十分大なる ν に対して, $S^\nu := \operatorname{con}(\{z_0, z_1, \cdots, z_m\})$ は単体. また有限個を除くすべての ν に対して 4.15 によって $\mathbf{0} \in \operatorname{int} S^\nu$. したがって, 有限個を除くすべての ν に対して $\sum_{i=1}^m \lambda_i = 1$ なる $\lambda_0^\nu, \lambda_1^\nu, \cdots, \lambda_m^\nu > 0$ が (一意に) 存在して

$$\mathbf{0} = \lambda_0^\nu z_0^\nu + \lambda_1^\nu z_1^\nu + \cdots + \lambda_m^\nu z_m^\nu.$$

2.28(f) によって $(\mathbf{0} \longrightarrow \mathbf{0})$, $\lambda_i^\nu \longrightarrow \lambda_i$ $(i = 0, 1, \cdots, m)$. ここで,

$$\mathbf{0} = \lambda_0 z_0 + \lambda_1 z_1 + \cdots + \lambda_m z_m$$

for $\lambda_0, \lambda_1, \cdots, \lambda_m > 0$.

それと同時に, $\bar{z}^\nu \longrightarrow \mathbf{0}$ であった. 原点をその内点に持つ有界な領域 B を考えたときに有限個を除いたすべての ν に対して、

$$\bar{z}^\nu \in B \subseteq S^\nu$$

となる (4.15) ので, これらの ν に対して,

$$\bar{z}^\nu = \bar{\lambda}_0^\nu z_0^\nu + \bar{\lambda}_1^\nu z_1^\nu + \cdots + \bar{\lambda}_m^\nu z_m^\nu, \text{ for } \bar{\lambda}_0^\nu, \bar{\lambda}_1^\nu, \cdots, \bar{\lambda}_m^\nu > 0 \text{ with } \sum_{i=0}^m \bar{\lambda}_i^\nu = 1.$$

ここで, 2.28(f) より $\bar{\lambda}_i^\nu \longrightarrow \lambda_i$ $(i = 0, 1, \cdots, m)$.

$$\theta^\nu := \min\{1, \lambda_0^\nu/\bar{\lambda}_0^\nu, \lambda_1^\nu/\bar{\lambda}_1^\nu, \cdots, \lambda_m^\nu/\bar{\lambda}_m^\nu\}$$

とすると, $0 < \theta^\nu \le 1$ かつ $\theta^\nu \longrightarrow 1$. よって, $\mu_i^\nu := \lambda_i^\nu - \theta^\nu \bar\lambda_i^\nu$ に対して, $0 \le \mu_i^\nu \longrightarrow 0$. また, $\sum_{i=0}^m \mu_i^\nu + \theta^\nu = 1$. ゆえに, X^ν と D^ν の凸性によって

$$X^\nu \ni \sum_{i=0}^m \mu_i^\nu x_i^\nu + \theta^\nu \bar x^\nu =: x^\nu \longrightarrow \bar x \quad \text{かつ} \quad D^\nu \ni \sum_{i=0}^m \mu_i^\nu u_i^\nu + \theta^\nu \bar u^\nu =: u^\nu \longrightarrow \bar u.$$

ところで,

$$\begin{aligned} L^\nu(x^\nu) - u^\nu &= \sum_{i=0}^m \mu_i^\nu L^\nu(x_i^\nu) + \theta^\nu L^\nu(\bar x^\nu) - \sum_{i=0}^m \mu_i^\nu u_i^\nu - \theta^\nu \bar u^\nu && (5)\\ &= \sum_{i=0}^m \mu_i^\nu (L^\nu(x_i^\nu) - u_i^\nu) + \theta^\nu (L^\nu(\bar x^\nu) - \bar u^\nu) && (6)\\ &= \sum_{i=0}^m \mu_i^\nu z_i^\nu + \theta^\nu \bar z^\nu && (7)\\ &= \sum_{i=0}^m (\lambda_i^\nu - \theta^\nu \lambda_i^\nu) z_i^\nu + \theta^\nu (\sum_{i=0}^m \bar\lambda_i^\nu z_i^\nu) && (8)\\ &= \sum_{i=0}^m \lambda_i^\nu z_i^\nu && (9)\\ &= \mathbf{0} && (10) \end{aligned}$$

だから, $x^\nu \in C^\nu$. □

系 (4.32(a)): *For linear mapping* $L^\nu, L\colon \mathbb{R}^n \to \mathbb{R}^m$ *and convex sets* $D^\nu \longrightarrow D$, *if* D *and* $\mathrm{Im}L$ *cannot be separated, then* $(L^\nu)^{-1}(D^\nu) \longrightarrow L^{-1}(D)$.

(証明) Take $X^\nu := \mathbb{R}^n$. □

系 (4.32(b)): *For matrices* $A^\nu \longrightarrow A$ *in* $\mathbb{R}^{m\times n}$ *and vectors* $b^\nu \longrightarrow b$ *in* $\mathbb{R}^m$, *if* A *has full rank* m, *then* $\{x \mid A^\nu x = b^\nu\} \longrightarrow \{x \mid Az = b\}$.

(証明) Set $D^\nu = \{b^\nu\}$, $D = \{b\}$ in (a). A のランク $= m$ ならどんなベクトルも $\mathrm{Im}(A)$ から分離されないけど、この条件は強すぎる？□

系 (4.32(c)): *For convex sets* $C_1^\nu, C_2^\nu \subseteq \mathbb{R}^n$, *the inclusion* $\liminf_\nu (C_1^\nu \cap C_2^\nu) \supseteq (\liminf_\nu C_1^\nu) \cap (\liminf_\nu C_2^\nu)$ *holds if the convex sets (4.15)* $\liminf_\nu C_1^\nu$ *and* $\liminf_\nu C_2^\nu$ *cannot be separated. Indeed,*

$$C_1^\nu \longrightarrow C_1, C_2^\nu \longrightarrow C_2 \Longrightarrow C_1^\nu \cap C_2^\nu \longrightarrow C_1 \cap C_2$$

as long as C_1 *and* C_2 *cannot be separated.*

(証明) $X^\nu := C_1^\nu$, $D^\nu := C_2^\nu$, $L := \mathrm{I}$ とする.
(前半:) 定理 4.32 の前半において, $X := \liminf_\nu X^\nu$, $D := \liminf_\nu D^\nu$ とおけば、

$$C^\nu = C_1^\nu \cap C_2^\nu,\ C = (\liminf_\nu X^\nu) \cap (\liminf_\nu D^\nu)$$

なので、主張が成り立つ。
(後半:) そのまんま. □

練習 (4.33): *For sequences of convex sets* $C_i^\nu \longrightarrow C_i$ $(i = 1, \cdots, q)$ *in* $\mathbb{R}^n$ *one has*

$$C_1^\nu \cap \cdots \cap C_q^\nu \longrightarrow C_1 \cap \cdots \cap C_q$$

if none of the limit sets C_i *cannot be separated from the intersection* $\bigcap_{k=1,k\neq i}^q C_k$ *of the others.*

(証明) q に関する帰納法による. $q = 2$ のときは, 4.32(c) である.
$C_i^\nu \longrightarrow C_i$ $(i = 1, \cdots, q)$ として, 各 $i = 1, \cdots, q$ に対して, C_i は $\bigcap_{k=1,k\neq i}^q C_k$ から分離されないとする. すると, 各 $i = 1, \cdots, q-1$ に対して, C_i は $\bigcap_{k=1,k\neq i}^{q-1} C_k$ から分離されない. ゆえに, 帰納法の仮定によって,

$$C_1^\nu \cap \cdots \cap C_{q-1}^\nu \longrightarrow C_1 \cap \cdots \cap C_{q-1}.$$

また, C_q は $\bigcap_{k=1}^{q-1} C_k$ から分離されないから,

$$(C_1^\nu \cap \cdots \cap C_{q-1}^\nu) \cap C_q^\nu \longrightarrow (C_1 \cap \cdots \cap C_{q-1}) \cap C_q.$$

これは言いたいことであった. □