Bisubmodular Polyhedra and Bidirected Graphs

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Abstract

We consider a "well-behaved" class of polyhedra characterized by the validity of a simple procedure called greedy algorithm for solving linear programming problems over them. F. D. J. Dunstan and D. J. A. Welsh first considered a general class of polyhedra for which a greedy algorithm works. The class of polyhedra is exactly that of bisubmodular polyhedra determined by bisubmodular functions. However, this relationship between the greedy algorithm and bisubmodularity was pointed out only recently by Chandrasekaran and Kabadi. The study of bisubmodular systems is important since bisubmodular systems give a unifying treatment for several existing models which are generalizations of the concept of matroid, and there exist interesting proper examples of bisubmodular systems such as the matchable subsets of an undirected graph and the nonsingular principal submatrices of a skew symmetric matrix.

For a finite set V let $\mathcal{F} \subseteq 3^V$ be a family of ordered pairs of disjoint subsets of V that is closed with respect to the reduced union $\sqcup$ and the intersection $\sqcap$. Also let f be a bisubmodular function on the $\{\sqcup, \sqcap\}$-closed family $\mathcal{F}$. We call the pair $(\mathcal{F}, f)$ a bisubmodular system on V if $(\emptyset, \emptyset) \in \mathcal{F}$, $f(\emptyset, \emptyset) = 0$ and $\mathcal{F}$ spans V. The bisubmodular polyhedron associated with a bisubmodular system $(\mathcal{F}, f)$ on V is given by

$$\mathrm{P}_*(f) = \{x \mid x \in \mathbf{R}^V, \forall (X, Y) \in \mathcal{F} \colon x(X, Y) \leq f(X, Y)\},$$

where $x(X, Y) = x(X) - x(Y)$ and $x(X) = \sum_{v \in X} x(v)$.

In Chapter 1 we introduce bisubmodular systems by giving a brief historical survey and some examples of bisubmodular systems. The organization of this dissertation is described along with the motivation of the study in each subsequent chapter. Also, we discuss preliminaries from lattice theory, graph theory, polyhedral theory and submodular system.

In Chapter 2 we describe elementary notions of $\{\sqcup, \sqcap\}$-closed families, bisubmodular systems and bidirected graphs and give fundamental results which are needed in the later chapters.

In Chapter 3 we consider some combinatorial optimization problems on bidirected graphs and networks. Bidirected graphs and networks are important in studies of bisubmodular systems. The cut functions of bidirected networks form an important class of bisubmodular functions. We show that the set of the boundaries of a bidirected network is the bisubmodular polyhedron associated with its cut function. On the other hand, we

associate, with each point $x \in \mathrm{P}_*(f)$, a bidirected graph $G(x)$ called an exchangeability graph. The exchangeability graph $G(x)$ represents the feasible directions at x in $\mathrm{P}_*(f)$ and the Hasse diagram of $G(x)$ represents the extreme directions at x. We consider how the Hasse diagram of a bidirected graph can be computed efficiently. Apart from applications to bisubmodular systems, we also consider generalizations of some classical network flow problems to bidirected networks. We show that a minimum cost circulation problem on a bidirected graph can be reduced to that on a directed graph. The maximum flow problem and the minimum cut problem are also reduced to the corresponding problems on ordinary directed graphs and the well-known maxflow-mincut theorem is generalized to bidirected networks. The minimum-weight ideal problem is considered as well.

When $x \in \mathrm{P}_*(f)$ is an extreme point of $\mathrm{P}_*(f)$, $G(x)$ defines a signed poset, an acyclic and transitively closed bidirected graph. V. Reiner has introduced the concept of signed poset and showed the so-called signed Birkhoff theorem that is a signed analogue of the well-known Birkhoff theorem. In Chapter 4 we show that there exists a one-to-one correspondence between the set of all the simple and spanning $\{\sqcup, \sqcap\}$-closed families $\mathcal{F} \subseteq 3^V$ on V and the set of all the signed posets $\mathcal{P}$ on V such that each such $\mathcal{F}$ is the set of all the ideals of the corresponding signed poset $\mathcal{P}$. This new observation enriches the signed Birkhoff theorem of V. Reiner. The result obtained in this chapter gives an important basis for developing a theory of bisubmodular functions and associated polyhedra.

In Chapter 5 we examine structures of bisubmodular polyhedra in terms of signed posets and exchangeability graphs. We give a characterization of extreme points together with an $\mathrm{O}(|V|^2)$ algorithm for discerning whether a given point is an extreme point, where we assume a function evaluation oracle for the bisubmodular function. The algorithm also determines the signed poset structure associated with the given point if it is an extreme point. We examine the greedy algorithm over possibly unbounded bisubmodular polyhedra and show an optimality condition in terms of an exchangeability graph. Characterizations of faces and their dimensions are given and the adjacency relation of extreme points is also provided in terms of the Hasse diagrams of the associated signed posets. Moreover, we investigate the connectivity and the decomposition of a bisubmodular system into connected components.

Simple necessary and sufficient conditions for a function to be bisubmodular are given in Chapter 6.

The another aim of this dissertation is to consider systems of inequalities described in terms of bidirected graphs.

An inequality in n variables $x_1, \cdots, x_n$ is of degree-two if it is either $x_i + x_j \leq 1$,

$-x_i - x_j \leq -1$ or $x_i - x_j \leq 0$ for some $i, j = 1, \cdots, n$. In many combinatorial optimization problems the feasible solutions can be described as the 0-1 solutions of degree-two constraints. In Chapter 7 we consider the LP relaxation of the 0-1 solutions of degree-two inequalities, which we call a fractional degree-two polytope. We show that any fractional degree-two polytope is isomorphic to an ideal polytope of a bidirected graph. The isomorphism shows an interesting relationship between the linear programs over fractional degree-two polytopes and the minimum-weight ideal problems for bidirected graphs. Also, any fractional degree-two polytope is shown to be half-integral and we give characterizations for a fractional degree-two polytope to be integral.

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Contents

Chapter 1

Introduction

In Section 1.1 we shortly review a historical aspect of bisubmodular systems and the organization of this dissertation is described along with the motivation of the study in each subsequent chapter. In Section 1.2 some examples of bisubmodular systems are in order. Also, we discuss preliminaries from lattice theory, graph theory, polyhedral theory and submodular system in Section 1.3.

1.1. Introduction

In this dissertation we consider a "well-behaved" class of polyhedra characterized by the validity of a simple procedure called greedy algorithm for solving linear programming problems over them. In 1973 F. D. J. Dunstan and D. J. A. Welsh [27] first considered a general class of polyhedra for which a greedy algorithm works. The class of polyhedra is exactly that of bisubmodular polyhedra. A bisubmodular polyhedron is defined by a system of linear inequalities with a $\{0, \pm 1\}$-coefficient matrix and a right-hand side expressed by a bisubmodular function. The study of bisubmodular systems is important since bisubmodular systems give a unifying treatment for several existing models which are generalizations of the concept of matroid, and there exist interesting proper examples of bisubmodular systems such as the matchable subsets of an undirected graph and the nonsingular principal submatrices of a skew symmetric matrix.

The concept of matroid was introduced by H. Whitney [84] as a combinatorial abstraction of linear dependence. In the recent decades matroid theory played an important role in combinatorics and combinatorial optimization and have a lot of applications to practical engineering problems (see [47], [55], [72]).

In 1970 J. Edmonds [29] combined matroid theory with polyhedral combinatorics, showing that the polyhedron given by the convex hull of the set of characteristic vectors of the independent sets of any matroid is described by linear inequalities with $\{0, 1\}$-coefficients and the rank function of the matroid in the right-hand. In [29], Edmonds also extended the concept of a matroid to a class of polyhedra called polymatroid.

Following the work of Edmonds [29], several generalizations of the concept of matroid and polymatroid have been proposed. Based on the observation that the nice properties of (poly-)matroids are due to the submodularity properties of the rank functions, S. Fujishige introduced the concept of submodular system [34] (see also [36]). This generalization is worth noting since submodular systems significantly widen the applicability. Especially, submodular systems recently have received considerable attentions in scheduling theory (see [71]). A. Frank's generalized (poly-)matroid [32] is a generalization of submodular system. However, every generalized polymatroid is a projection of a base polyhedron associated with a submodular system, as was noted by Fujishige [35].

R. Chandrasekaran and S. N. Kabadi [21] introduced the concept of pseudomatroid as a common generalization of matroid and generalized matroid. The same or similar concepts as pseudomatroid were independently considered by A. Bouchet ([17], [18]) as delta-matroid, A. Dress and T. Havel [23] as metroid. Chandrasekaran and Kabadi [21] defined bisubmodular polyhedron as the polyhedral extension of pseudomatroids, and then, the relationship between the greedy algorithm and bisubmodularity was pointed out without mentioning the result of Dunstan and Welsh [27]. It should be noted that the same concept as bisubmodular polyhedron was independently considered by M. Nakamura [62] as universal polymatroid.

Equipped with the signed Birkhoff theorem ([73], [4]), the author and the colleagues made significant contributions to the study of bisubmodular systems and associated polyhedra, of which this dissertation consists.

It should be noted here that the term, bisubmodular, was first used by A. Schrijver [76] in a way slightly different from the one defined in this dissertation. Also, bisubmodular functions defined here are called generalized submodular in [21], directed submodular in [67]. Our usage is the same as the one in [19] and [68].

Denote by 3^V the set of all the ordered pairs of disjoint subsets of a nonempty finite set V, i.e., $3^V = \{(X,Y) \mid X, Y \subseteq V, X \cap Y = \emptyset\}$. We have two binary operations, $\sqcup$ and $\sqcap$, on 3^V defined as

$$\begin{aligned}(X_1,Y_1) \sqcup (X_2,Y_2) &= ((X_1 \cup X_2) - (Y_1 \cup Y_2), (Y_1 \cup Y_2) - (X_1 \cup X_2)),\\ (X_1,Y_1) \sqcap (X_2,Y_2) &= (X_1 \cap X_2, Y_1 \cap Y_2)\end{aligned}$$

for each $(X_1,Y_1),(X_2,Y_2) \in 3^V$. A bisubmodular system is the pair $(\mathcal{F}, f)$ of a $\{\sqcup,\sqcap\}$-closed family $\mathcal{F} \subseteq 3^V$ and a bisubmodular function f on $\mathcal{F}$, i.e., for any $(X_1,Y_1),(X_2,Y_2) \in \mathcal{F}$ we have

$$f(X_1,Y_1) + f(X_2,Y_2) \geq f((X_1,Y_1) \sqcup (X_2,Y_2)) + f((X_1,Y_1) \sqcap (X_2,Y_2)),$$

where we assume $(\emptyset, \emptyset) \in \mathcal{F}$, $f(\emptyset, \emptyset) = 0$ and $\mathcal{F}$ spans V. Associated with $(\mathcal{F}, f)$ the bisubmodular polyhedron $\mathrm{P}_*(f)$ is defined as follows.

$$\mathrm{P}_*(f) = \{x \mid x \in \mathbf{R}^V, \forall (X, Y) \in \mathcal{F}\colon x(X, Y) \leq f(X, Y)\},$$

where $x(X, Y) = x(X) - x(Y)$ for any $(X, Y) \in 3^V$.

In Chapter 2 we describe elementary notions of $\{\sqcup, \sqcap\}$-closed families, bisubmodular systems and bidirected graphs and give fundamental results which are needed in the later chapters.

A bidirected graph is a graph where each arc has two end-vertices which are either two heads, two tail, or one head and one tail. In Chapter 3 we consider some combinatorial optimization problems on bidirected graphs and networks, where we apply the reduction technique used in [10] and [11]. The importance of bidirected graphs and networks in studies of bisubmodular systems are explained in two ways. First, they provide us with a class of bisubmodular functions, namely, cut functions of bidirected networks. Moreover, the sets of the boundaries of flows in bidirected networks are described as bisubmodular polyhedra. It is interesting to see that these polyhedra are projections of certain base polyhedra. Secondly, we associate, with each point $x \in \mathrm{P}_*(f)$, a bidirected graph $G(x)$, which represents the feasible directions at x in $\mathrm{P}_*(f)$. We will see that the extreme directions at x can be represented by the Hasse diagram of the exchangeability graph $G(x)$. An efficient algorithm for deriving the Hasse diagram of a bidirected graph is given. Apart from applications to bisubmodular systems, we also consider generalizations of some classical network flow problems to bidirected networks. We show that a minimum cost circulation problem on a bidirected graph can be reduced to that on a directed graph. The maximum flow problem and the minimum cut problem are also reduced to the corresponding problems on ordinary directed graphs and the well-known maxflow-mincut theorem is generalized to bidirected networks. The minimum-weight ideal problem is considered as well. Chapter 3 is based on the manuscript [2] and partially on the collaborated work [8] with S. Fujishige and T. Naitoh.

We will see that a point $x \in \mathrm{P}_*(f)$ is an extreme point of $\mathrm{P}_*(f)$ if and only if $G(x)$ defines a signed poset, i.e., an acyclic and transitively closed bidirected graph. V. Reiner [73] has introduced the concept of signed poset and showed the so-called signed Birkhoff theorem that is a signed analogue of the well-known Birkhoff theorem on the relationship between the set of ideals of a poset and a distributive lattice. In Chapter 4 we show that there exists a one-to-one correspondence between the set of all the simple and spanning $\{\sqcup, \sqcap\}$-closed families $\mathcal{F} \subseteq 3^V$ on V with $(\emptyset, \emptyset) \in \mathcal{F}$ and the set of all the signed posets $\mathcal{P}$ on V such that each such $\mathcal{F}$ is the set of all the ideals of the corresponding signed poset $\mathcal{P}$. This new observation enriches the signed

Birkhoff theorem of V. Reiner. Representations of general $\{\sqcup, \sqcap\}$-closed families and simplifications of bisubmodular systems are also considered. The result obtained in this chapter gives an important basis for developing a theory of bisubmodular functions and associated polyhedra. Chapter 4 is based on the collaborated work [5] with S. Fujishige.

Chapter 5 is also based the collaboration [5] with S. Fujishige. We examine structures of bisubmodular polyhedra in terms of signed posets and exchangeability graphs. We give characterizations of boundedness and pointedness of bisubmodular polyhedra and also give a characterization of extreme points together with an $\mathrm{O}(|V|^2)$ algorithm for discerning whether a given point is an extreme point, where we assume a function evaluation oracle for the bisubmodular function. The algorithm also determines the signed poset structure associated with the given point if it is an extreme point. We reveal the adjacency relation of extreme points by means of the Hasse diagrams of the associated signed posets. Moreover, we investigate the connectivity and the decomposition of a bisubmodular system into its connected components.

Simple necessary and sufficient conditions for a function to be bisubmodular are given in Chapter 6. The chapter is based on the joint work [7] with S. Fujishige and T. Naitoh.

The another aim of this dissertation is to consider combinatorial optimization problems associated with bidirected graphs. Some of them are considered in Chapter 3. The whole of Chapter 7 is devoted to analyze a class of polyhedra described by means of bidirected graphs.

An inequality in n variables $x_1, \cdots, x_n$ is of degree-two if it is either $x_i + x_j \leq 1$, $-x_i - x_j \leq -1$ or $x_i - x_j \leq 0$ for some $i, j = 1, \cdots, n$. In many combinatorial optimization problems the feasible solutions can be described as the 0-1 solutions of degree-two constraints. The stable sets, the node covers, the order ideals of a directed graph are described as the 0-1 solutions of systems of degree-two inequalities. Other examples of the 0-1 solutions of degree-two inequalities can be found in many contexts, such as quadratic Boolean equations [42], linear and quadratic 0-1 programming [41], and logic [75]. It should also be noted that a system of degree-two constraints is, in disguise, a *complete set of implicants* with their lengths at most two (see [45], [44]). We consider a relaxation of the 0-1 solutions of degree-two inequalities, namely, we consider the solution set of degree-two inequalities and the inequalities $0 \leq x_j \leq 1$ $(j = 1, \cdots, n)$, which we call a fractional degree-two polytope (FD2P). We show that any FD2P is isomorphic to an ideal polytope of a bidirected graph. The isomorphism shows that an interesting relationship between the linear programs over FD2P's and the minimum-weight ideal problems for bidirected graphs. Finally, we give characterizations for an

FD2P to be integral and apply them to some set systems characterized by complete sets of implicants. Chapter 7 is based on the paper [3].

Finally in Chapter 8 we conclude the dissertation by presenting related topics and future research topics.

Throughout this dissertation, V stands for a nonempty finite set and $\mathbf{R}$ is the set of reals and $\mathbf{Z}$ the set of integers. $\mathbf{R}_+$ and $\mathbf{Z}_+$ denote the sets of nonnegative elements of $\mathbf{R}$ and $\mathbf{Z}$, respectively. The set of all the mappings from a set A to a set B is denoted by B^A. For any finite set X we denote by $|X|$ the cardinality of X.

1.2. Introductory Examples of Bisubmodular Systems

As is mentioned in Section 1.1, the class of bisubmodular systems is broad enough to include many existing models such as (poly-)matroids and submodular systems. Nevertheless, there exist important combinatorial structures which are bisubmodular systems but are not any other existing structures. In this section, we shortly review such proper applications of bisubmodular systems.

a. Delta-matroids

The pair $(V, \mathcal{M})$ of a finite set V and a family $\mathcal{M}$ of subsets of V is called a *delta-matroid* if $\mathcal{M}$ satisfies the following *symmetric exchange axiom* (SEA).

(SEA) For any $W_1, W_2 \in \mathcal{M}$ and $v \in W_1 \Delta W_2$ there exists $w \in W_1 \Delta W_2$ such that $W_1 \Delta \{v, w\} \in \mathcal{M}$,

where Δ denotes the symmetric difference operator. The structure of pseudomatroid ([21]) is equivalent to that of delta-matroid. However, the term "delta-matroid" is nowadays commonly used for the same object. A metroid [23] is merely a delta-matroid $(V, \mathcal{M})$ with $\emptyset \in \mathcal{M}$ (see [20]). The structure of ditroids [67] is similar to but weaker than that of delta-matroid.

Both the family of the bases and the family of the independent subsets of a matroid satisfies (SEA). Moreover, interesting examples exist as follows.

Matching delta-matroids ([18])**:** Let $G = (V, E)$ be an undirected graph. A subset W of V is called *matchable* if there exists a perfect matching in the subgraph induced by W. Let $\mathcal{M}$ be the set of all the matchable subset of V. Then, $\mathcal{M}$ satisfies the symmetric exchange axiom. ■

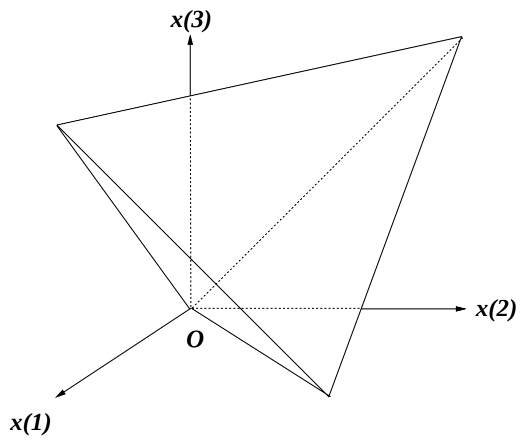

Figure 1.1: The matching delta-matroid polyhedron of K_3.

Nonsingular principal submatrices of a skew-symmetric matrix ([19])**:** Let $A = (a_{ij})_{i,j \in V}$ be a skew-symmetric matrix over any field with its row and column index sets V. For each subset W of V denote by A_W the submatrix of A defined by $A_W = (a_{ij})_{i,j \in W}$. Define $\mathcal{M} = \{W \mid W \subseteq V, A_W \text{ is nonsingular}\}$. Then, $(V, \mathcal{M})$ is a delta-matroid. ■

The *rank function* $f : 3^V \to \mathbf{Z}$ of a delta-matroid is defined as

$$f(X, Y) = \max\{|X \cap W| - |Y \cap W| \mid W \in \mathcal{M}\} \quad ((X, Y) \in 3^V). \tag{1.1}$$

Theorem 1.1 ([51], [21]): *Suppose that* $(V, \mathcal{M})$ *is a delta-matroid and* f *is its rank function. Then,* $(3^V, f)$ *is a bisubmodular system on* V *and the convex hull of the characteristic vectors of all the elements of* $\mathcal{M}$ *is given by the bisubmodular polyhedron associated with* $(3^V, f)$. □

b. Jump systems

A. Bouchet and W. H. Cunningham [19] have recently introduced the concept of jump system. For any $x, y \in \mathbf{Z}^V$ a *step* s *from* x *to* y is a $\{0, \pm 1\}$-vector such that $\sum_{v \in V} |x(v) + s(v) - y(v)| = \sum_{v \in V} |x(v) - y(v)| - 1$. We denote by $\mathrm{St}(x, y)$ the set of all the steps from x to y. A *jump system* on V is a pair $(V, \mathcal{J})$ of a nonempty finite set V and a nonempty $\mathcal{J} \subseteq \mathbf{Z}^V$ which satisfies the 2-*step axiom*:

(2-SA) For any $x, y \in \mathcal{J}$ and $s \in \mathrm{St}(x, y)$ with $x + s \notin \mathcal{J}$ there exists $t \in \mathrm{St}(x + s, y)$ such that

$$x + s + t \in \mathcal{J}. \tag{1.2}$$

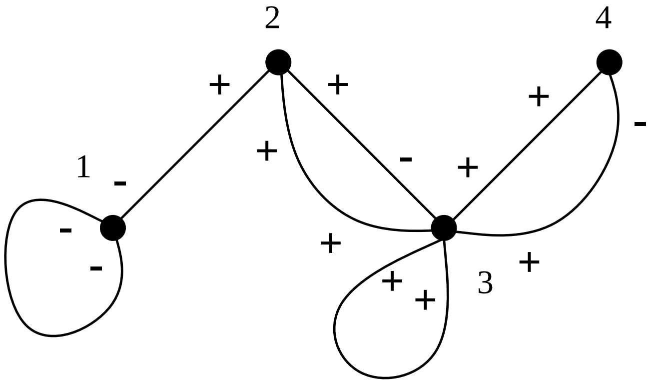

Figure 1.2: A bidirected graph.

A delta-matroid is a jump system if we identify a subset W of V with its characteristic vector $\chi_W \in \mathbf{R}^V$. Indeed, provided $\mathcal{J} \subseteq \{0,1\}^V$, $(V, \mathcal{J})$ is a jump system if and only if $(V, \mathcal{J})$ is a delta-matroid ([19]).

Proposition 1.2 ([19]): *For a jump system $(V, \mathcal{J})$ the convex hull of $\mathcal{J}$ coincides with a bisubmodular polyhedron in $\mathbf{R}^V$.* □

c. Boundaries of bidirected flows

A *bidirected graph* $G = (V, A; \partial)$ is a graph with a vertex set V, an arc set A and a boundary operator $\partial \colon A \to \mathbf{Z}^V$, where for each arc $a \in A$ there exist $v, w \in V$ (called *end-vertices* of a) such that one of the following three holds:

(1) $\partial a = v + w$ (arc a has two tails at v and w),
(2) $\partial a = -v - w$ (arc a has two heads at v and w),
(3) $\partial a = v - w$ (arc a has a tail at v and a head at w).

Here, each $\partial a \in \mathbf{Z}^V$ is represented by an element of a free module with a base V. If $v = w$ in (1)$\sim$(3), then arc a is called a *selfloop*. For simplicity we do not allow any selfloop of type (3) in the following. See Figure 1.2 for an example of a bidirected graph.

We call a pair $\mathcal{N} = (G = (V, A; \partial), c)$ of a bidirected graph $G = (V, A; \partial)$ and a function $c \colon A \to \mathbf{R}_+$ a *bidirected network*. The function c is called a *capacity function*. The *cut function* $\kappa_c \colon 3^V \to \mathbf{R}$ of a bidirected network $\mathcal{N}$ is defined as

$$\kappa_c(X, Y) = \sum\{\langle \partial a, (X, Y)\rangle c(a) \mid a \in A, \langle \partial a, (X, Y)\rangle > 0\} \quad ((X, Y) \in 3^V), \qquad (1.3)$$

where (X, Y) and ∂a are regarded as vectors in $\mathbf{R}^V$ and $\langle \cdot, \cdot \rangle$ stands for the (canonical) inner product.

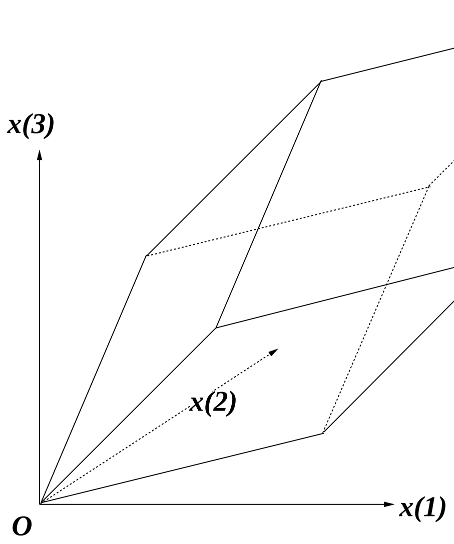

Figure 1.3: The degree-sequence polyhedron of K_3.

Proposition 1.3: *$\kappa_c\colon 3^V \to \mathbf{R}$ is a bisubmodular function.* □

For a bidirected network $\mathcal{N} = (G = (V, A), c)$, a function $\varphi\colon A \to \mathbf{R}_+$ is called a (*feasible*) *flow* in $\mathcal{N}$ if $0 \le \varphi(a) \le c(a)$ for all $a \in A$. For any function $\varphi\colon A \to \mathbf{R}$ the *boundary* $\partial\varphi\colon V \to \mathbf{R}$ of φ is given by

$$\partial\varphi = \sum\{\varphi(a)\partial a \mid a \in A\}, \tag{1.4}$$

where ∂a is considered as a vector in $\mathbf{R}^V$.

Proposition 1.4: *Let $\partial\Phi$ be the set of all the boundaries of feasible flows in a bidirected network $\mathcal{N} = (G = (V, A; \partial), c)$. We have*

$$\partial\Phi = \mathrm{P}_*(\kappa_c). \tag{1.5}$$

□

In Section 3.2 we give the proofs of Lemma 1.3 and Proposition 1.4 in slightly generalized forms.

For example, let us consider the complete graph $\mathrm{K}_3 = (V, A)$ on vertex set $V = \{1, 2, 3\}$. We regard K_3 as a bidirected graph by considering each arc of K_3 is positively incident to its end-vertices. Define a capacity function $c\colon A \to \mathbf{R}_+$ as $c(a) = 1$ for every arc a of K_3. Then, the associated cut function $\kappa_c\colon 3^V \to \mathbf{R}_+$ is given as

$$\kappa_c(X, Y) = 2|\Gamma(X)| + |\Delta(X, V - (X \cup Y))| \quad ((X, Y) \in 3^V), \tag{1.6}$$

where $\Gamma(X)$ is the set of edges both of which end-vertices are in X and $\Delta(X, V-(X\cup Y))$ the set of edges which connect a vertex in X and a vertex in $V-(X\cup Y)$. The polyhedron $\mathrm{P}_*(k_c)$ is the convex hull of the degree vectors of the spanning subgraphs of K_3. See Figure 1.3.

1.3. Fundamental Definitions and Preliminaries

In this section we review fundamental notions from lattice theory, inequality systems, graph theory and submodular systems, here we follow the terminology, notations and definitions given in Fujishige's book [36]. This section serves as a reference for the later chapters. For more extensive treatments see, e.g., [14] for lattice theory, [1] for theory of network flow, [77] for polyhedral theory and [36] for submodular system.

a. Sets, posets and lattices

When X is a subset of a set Y, we write $X \subseteq Y$, and when X is a proper subset of Y (i.e., $X \subseteq Y$ and $X \neq Y$), we write $X \subset Y$. For two sets X and Y, when $X \cap Y = \emptyset$, we say X and Y are *disjoint*. For any two sets X and Y $X - Y$ stands for the set difference. The *characteristic vector* of a subset A of an underlying set V is the mapping $\chi_A \colon V \to \{0,1\}$ such that $\chi_A(v) = 1$ for $v \in A$ and $\chi_A(v) = 0$ for $v \in V - A$. We write χ_v as the characteristic vector of a singleton set $\{v\} \subseteq V$, instead of $\chi_{\{v\}}$.

A set Π of disjoint nonempty subsets of a set V is called a *partition* of V if $\bigcup_{F\in\Pi} F = V$.

A binary relation $\preceq$ on a set A is called a *partial order* if it satisfies the following three conditions:

(i) (reflexivity) $\forall a \in A\colon a \preceq a$,

(ii) (antisymmetry) $a \preceq b, b \preceq a \Longrightarrow a = b$,

(iii) (transitivity) $a \preceq b, b \preceq c \Longrightarrow a \preceq c$.

We call the pair $(A, \preceq)$ a *partially ordered set* (or simply a *poset*). Also, we often call A itself a poset when the underlying partial order $\preceq$ is implicitly assumed. If $a \preceq b$ and $a \neq b$, we write $a \prec b$.

For a poset $\mathcal{P} = (A, \preceq)$, a subset B of A is called a *(lower) order ideal* of $\mathcal{P}$ if $x \preceq y \in B$ implies $x \in B$. Also, a subset B of A is called an *upper order ideal* of $\mathcal{P}$ if $B \ni x \preceq y$ implies $y \in B$.

For two distinct elements x, y of a poset $\mathcal{P} = (A, \preceq)$, if there is no element $z \in A$ such that $x \prec z \prec y$, we say y *covers* x.

Let $\mathcal{P} = (A, \preceq)$ be a poset. An element $a \in A$ is an *upper* (*lower*) *bound* of a subset $B \subseteq A$ if $b \preceq a$ ($b \succeq a$) for all $b \in B$. If a is an upper (lower) bound of B and $a \in B$, a is called the *maximum* (*minimum*) element of B. The maximum (or minimum) element, if it exists, is unique. If the set of upper (lower) bounds of B has the minimum (maximum) element b_* (b^*), we call b_* (b^*) the *least upper bound* (*greatest lower bound*) of B and denote by $\sup B$ ($\inf B$). If for each $x, y \in A$ there exist $\sup\{x, y\}$ and $\inf\{x, y\}$ in A, then we call the poset a *lattice*, and we write $\sup\{x, y\}$ as $x \vee y$ and $\inf\{x, y\}$ as $x \wedge y$. These two operations, $\vee$ and $\wedge$, are called the *join* and the *meet*, respectively, and satisfy

(i) (idenpotency) $\forall x \in A\colon x \vee x = x,\ x \wedge x = x,$

(ii) (commutativity) $\forall x, y \in A\colon x \vee y = y \vee x,\ x \wedge y = y \wedge x,$

(iii) (associativity) $\forall x, y, z \in A\colon x \vee (y \vee z) = (x \vee y) \vee z,$
$$x \wedge (y \wedge z) = (x \wedge y) \wedge z,$$

(iv) (absorption) $\forall x, y \in A\colon x \wedge (x \vee y) = x,\ x \vee (x \wedge y) = x.$

Conversely, if a set A with two binary operations $\vee$ and $\wedge$ satisfies the above (i)$\sim$(iv), then, defining a binary relation $\preceq$ by "$x \preceq y \Longleftrightarrow x \vee y = y$ (or $x \wedge y = x$)", we have a lattice $(A, \preceq)$ whose binary operations $\vee$ and $\wedge$ are the given $\vee$ and $\wedge$. A lattice $(A, \preceq)$ with lattice operations $\vee$ and $\wedge$ is also expressed as $(A, \vee, \wedge)$. We often call A itself a lattice assuming the underlying partial order or lattice operations.

A lattice $(A, \preceq)$ is *complete* if for any nonempty subset B of A there exist both $\inf B$ and $\sup B$ in A.

For a lattice $(A, \preceq)$ with the minimum element O an element x is called an *atom* if it covers O.

A lattice $\mathcal{L} = (A, \vee, \wedge)$ is called *distributive* if it satisfies

(v) (distributivity) $\forall x, y, z \in A\colon x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),$
$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

A lattice $\mathcal{L} = (A, \vee, \wedge)$ with the minimum element O and the maximum element I is called *complemented* if for each $x \in A$ there exists $y \in A$ such that $x \wedge y = O$ and $x \vee y = I$. A lattice is called a *Boolean lattice* if it is distributive and complemented.

b. Graphs and networks

Let V and A be finite sets, where V is called a *vertex set* and A an *arc set*. Each $v \in V$ is called a *vertex* and each $a \in A$ is called an *arc*. Each arc $a \in A$ has its *initial end-vertex* $\partial^+ a$ and *terminal end-vertex* $\partial^- a$. We call $G = (V, A; \partial^+, \partial^-)$ a *(directed) graph*. We often denote such a directed graph by $G = (V, A)$.

Mappings $\delta^+ \colon V \to 2^A$ and $\delta^- \colon V \to 2^A$ are, respectively, defined by

$$\delta^+ v = \{a \mid a \in A, v = \partial^+ a\}, \tag{1.7}$$
$$\delta^- v = \{a \mid a \in A, v = \partial^- a\} \tag{1.8}$$

for each $v \in V$.

A *(directed) network* $\mathcal{N} = (G = (V, A), c)$ is a pair of a directed graph $G = (V, A)$ and a nonnegative function $c : A \to \mathbf{R}_+ \cup \{+\infty\}$ on A. The function c is called a *capacity function*. A function $\varphi : A \to \mathbf{R}_+$ is called a *feasible flow* in the directed network $\mathcal{N} = (G = (V, A), c)$ if it satisfies

$$\forall a \in A \colon 0 \le \varphi(a) \le c(a). \tag{1.9}$$

The *boundary* $\partial\varphi \colon V \to \mathbf{R}_+$ of a flow φ is a real-valued function on V defined as

$$\partial\varphi(v) = \sum_{a \in \delta^+ v} \varphi(a) - \sum_{a \in \delta^- v} \varphi(a) \quad (v \in V), \tag{1.10}$$

or equivalently as

$$\partial\varphi = \sum \{\varphi(a)(\chi_{\partial^+ a} - \chi_{\partial^- a}) \mid a \in A\}. \tag{1.11}$$

Let us denote the set of boundaries of flows in the directed network $\mathcal{N}$ by $\partial\Phi$. For each subset U of V, the arc set $\Delta^+(U) \subseteq A$ defined by

$$\Delta^+(U) = \{a \mid a \in A, \partial^+ a \in U, \partial^- a \in V - U\} \tag{1.12}$$

is called a *cut* of a directed graph $G = (V, A)$. The *cut function* $\kappa_c \colon 2^V \to \mathbf{R}_+ \cup \{+\infty\}$ of a directed network $\mathcal{N} = (G = (V, A), c)$ is defined by

$$\kappa_c(U) = \sum_{a \in \Delta^+(U)} c(a) \quad (U \subseteq V), \tag{1.13}$$

or equivalently by

$$\kappa_c(U) = \sum \{c(a) \mid a \in A, \langle \chi_{\partial^+ a} - \chi_{\partial^- a}, \chi_U \rangle > 0\} \quad (U \subseteq V). \tag{1.14}$$

For a directed graph $G = (V, A; \partial^+, \partial^-)$ a subsets J of V is called a *(lower) order ideal* of G if for each $a \in A$ $\partial^+ a \in J$ implies $\partial^- a \in J$. Note that $J \subseteq V$ is an order ideal of G if and only if $\langle \chi_{\partial^+ a} - \chi_{\partial^- a}, \chi_J \rangle \le 0$ holds for any $a \in A$.

c. Systems of linear inequalities and linear programming problems

Let $x_i \in \mathbf{R}^V$ $(i \in I)$, where I is a nonempty finite index set. For $\lambda_i \in \mathbf{R}$ $(i \in I)$ $\sum_{i\in I} \lambda_i x_i$ is called a *linear combination* (or simply *combination*) of x_i $(i \in I)$. A combination $\sum_{i\in I} \lambda_i x_i$ of x_i $(i \in I)$ is called an *affine* (respectively, *conical*) *combination* if $\sum_{i\in I} \lambda_i = 1$ (respectively, $\lambda_i \geq 0$ $(i \in I)$) holds. An affine and conical combination is called a *convex combination*. For a subset S of $\mathbf{R}^V$ the set of all the affine (respectively, conical and convex) combinations of elements of S is called the *affine* (respectively, *conical* and *convex*) *hull* of S and denoted by aff.(S) (respectively, cone(S) and conv.(S)). We define aff.$(\emptyset) = \emptyset$, cone$(\emptyset) = \{\mathbf{0}\}$ and conv.$(\emptyset) = \emptyset$. A subset $S \subseteq \mathbf{R}^V$ is called a *affine set* (respectively, *convex cone* and *convex set*) if each affine (respectively, conical and convex) combination of elements of S belongs to S, that is, aff.$(S) = S$ (respectively, cone$(S) = S$ and conv.$(S) = S$). We sometimes call a convex cone simply a *cone*.

A *convex polyhedron* (or simply *polyhedron*) P is the solution set of a finite system of inequalities:

$$\sum_{v\in V} a_i(v)x(v) \leq b_i \quad (i \in I), \tag{1.15}$$

where $a_i(v), b_i$ $(i \in I, v \in V)$ are real constants and I is a finite index set. Clearly, a polyhedron is a convex set. Note that different systems of linear inequalities may give the same polyhedron. We call (1.15) a *representation* of P.

The *dimension* of a polyhedron P is defined as the dimension of the affine hull of P.

A cone which is a polyhedron is called a *polyhedral cone*. The solution set of a finite system of homogeneous inequalities

$$\sum_{i\in I} a_i(v)x(v) \leq 0 \quad (i \in I) \tag{1.16}$$

is a polyhedral cone. The converse is also true, that is, each polyhedral cone is the solution set of a finite system of homogeneous inequalities. A cone $C \subseteq \mathbf{R}^V$ is *generated* by a_i $(i \in I)$ if C is the conical hull of vectors a_i $(i \in I)$. When I is finite, we say C is *finitely generated*. A cone is finitely generated if and only if it is polyhedral.

For a cone $C \subseteq \mathbf{R}^V$, the *polar cone* (or *dual cone*) C^* of C is defined as

$$C^* = \{x \mid x \in \mathbf{R}^V, \forall y \in C\colon \sum_{v\in V} x(v)y(v) \leq 0\}. \tag{1.17}$$

The dual cone of the polyhedral cone described by (1.16) is the conical hull of vectors a_i $(i \in I)$. Conversely, the dual cone of the cone finitely generated by a_i $(i \in I)$ is the cone described by (1.16).

For a convex polyhedron P described by (1.15) the *characteristic cone* of P is the solution set of the system of equations:

$$\sum_{v \in V} a_i(v)x(v) \leq 0 \quad (i \in I). \tag{1.18}$$

Also, the *lineality space* of a polyhedron P is the largest linear subspace contained in the characteristic cone of P. The lineality space of the polyhedron described by (1.15) is the solution set of the following system of homogeneous equations:

$$\sum_{v \in V} a_i(v)x(v) = 0 \quad (i \in I). \tag{1.19}$$

For a polyhedron P described by (1.15) and $x_0 \in P$ the *tangent cone* of P at x_0 is the cone defined as the solution set of the following system of homogeneous inequalities:

$$\sum_{v \in V} a_i(v)x(v) \leq 0 \quad (i \in I_=), \tag{1.20}$$

where $I_= = \{i \mid i \in I, \sum_{v \in V} a_i(v)x_0(v) = b_i\}$.

For any $I_0 \subseteq I$ denote by $P(I_0)$ the convex polyhedron described by

$$\begin{aligned} \sum_{v \in V} a_i(v)x(v) &= b_i \quad (i \in I_0), & (1.21)\\ \sum_{v \in V} a_i(v)x(v) &\leq b_i \quad (i \in I - I_0). & (1.22) \end{aligned}$$

Here, it should be noted that $P(I_0)$ may be empty. Define

$$\mathcal{F}(P) = \{P(I_0) \mid I_0 \subseteq I\} \cup \{\emptyset\}. \tag{1.23}$$

Each $F \in \mathcal{F}(P)$ is called a *face* of P. A face which is a proper subset of P is called a *proper face* of P. Faces of P are determined by P and are independent of the choice of a representation (1.15) of P.

Lemma 1.5: *The faces $\mathcal{F}(P)$ of a polyhedron P form a complete lattice with respect to set inclusion as the partial order. For each $F_1, F_2 \in \mathcal{F}(P)$ we have*

$$\begin{aligned} F_1 \vee F_2 &= \bigcap \{F \mid F \in \mathcal{F}(P),\ F \supseteq F_1 \cup F_2\}, & (1.24)\\ F_1 \wedge F_2 &= F_1 \cap F_2. & (1.25) \end{aligned}$$

□

The lattice $\mathcal{F}(P)$ is called the *face lattice* of P.

A zero-dimensional face is called a *vertex*. The point of a vertex is called an *extreme point*. A one-dimensional face is called an *edge*. Two vertices is said to be *adjacent* if they are contained in a common edge. An edge of a convex cone is called an *extreme ray*. An extreme ray of a polyhedron is defined as an extreme ray of its characteristic cone.

A polyhedron P is said to be *integral* if each nonempty minimal face of P contains an integral point. Also, a polyhedron P is *half-integral* if each nonempty minimal face of P contains a vector whose components are all integral multiples of $\frac{1}{2}$.

A polyhedron which has a vertex is called *pointed*. A polyhedron $P \subseteq \mathbf{R}^V$ is called a *bounded polyhedron* (or a *polytope*) if there exists a positive real M such that $\max\{|x(v)| \mid v \in V\} \leq M$ for all $x \in P$.

Theorem 1.6 ([81]): *A nonempty polyhedron is pointed if and only if its lineality space consists of the zero vector only. A nonempty polyhedron is bounded if and only if its characteristic cone is equal to the set of the zero vector.* □

Let us consider a *linear programming problem*:

$$(LP) \qquad \text{Maximize} \sum_{v \in V} w(v)x(v) \tag{1.26}$$

$$\text{subject to} \sum_{v \in V} a_i(v)x(v) \leq b_i \quad (i \in I), \tag{1.27}$$

where $a_i(v), b_i, w(v) \in \mathbf{R}$ $(i \in I, v \in V)$ are given constants and I is a finite index set. The *dual* linear programming problem of (P) is given by

$$(DLP) \qquad \text{Minimize} \sum_{i \in I} b_i \lambda_i \tag{1.28}$$

$$\text{subject to} \sum_{i \in I} a_i(v)\lambda_i = w(v) \quad (v \in V), \tag{1.29}$$

$$\lambda_i \geq 0 \quad (i \in I). \tag{1.30}$$

Problem (LP) is called a *primal* problem and is the dual of (DLP). A *feasible solution* of Problem (LP) (or (DLP)) is a solution of (1.27) (or (1.29) and (1.30)). We say Problem (LP) (or (DLP)) is *feasible* if the problem has a feasible solution. We say Problem (LP) (or (DLP)) is *unbounded* if the problem is feasible and the value of the objective function can be made arbitrarily large (or small). A feasible solution x_0 for (LP) is called an *optimal solution* for Problem (LP) if for any feasible solution x for (LP) we have $\sum_{v \in V} w(v)x(v) \leq \sum_{v \in V} w(v)x_0(v)$. If x_0 is an optimal solution for Problem (LP), the value $\sum_{v \in V} w(v)x_0(v)$ is called the *optimal value* for Problem (LP).

Theorem 1.7 (The duality theorem for linear programming): *If one of Problems* (LP) *and* (DLP) *has an optimal solution, then its dual problem also has an optimal solution and for any optimal solutions* x^* *of* (LP) *and* λ^* *of* (DLP) *we have*

$$\sum_{v \in V} w(v)x^*(v) = \sum_{i \in I} b_i \lambda_i^*. \tag{1.31}$$

Moreover, if one of the problems (LP) *and* (DLP) *is unbounded, then its dual problem is not feasible.* □

For a feasible problem (LP) a characterization of boundedness and an optimality condition are given in terms of the characteristic cone and the tangent cone, respectively.

Theorem 1.8: *Suppose that* P *is a nonempty polyhedron described by* (1.27).
(i) *Problem* (LP) *has a finite optimal solution if and only if* $w \in \mathbf{R}^V$ *is an element of the dual cone of the characteristic cone of the polyhedron.*
(ii) x_0 *is an optimal solution for Problem* (LP) *if and only if* w *is an element of the dual cone of the tangent cone of* P *at* x_0. □

d. Submodular systems

For any $S \subseteq V$ and $x \in \mathbf{R}^V$ let us denote $\sum_{v \in S} x(v)$ by $x(S)$. Suppose $|V| = n$.

Let $\mathcal{D}$ be a family of subsets of V which forms a distributive lattice under set union and intersection as the lattice operations, join and meet, i.e., for each $X, Y \in \mathcal{D}$ we have $X \cup Y, X \cap Y \in \mathcal{D}$. First, we notice the well-known theorem of Birkhoff on distributive lattices.

Theorem 1.9 (Birkhoff [14]): *For any distributive lattice* $\mathcal{D}$ *of subsets of a finite set* V *with* $\emptyset, V \in \mathcal{D}$, *there exists a unique poset* $\mathcal{P}(\mathcal{D}) = (\Pi(\mathcal{D}), \preceq)$ *such that*

(i) $\Pi(\mathcal{D})$ *is a partition of* V.
(ii) $X \in \mathcal{D}$ *if and only if* $X = \bigcup\{F \mid F \in J\}$ *for some order ideal* J *of* $\mathcal{P}(\mathcal{D})$. □

We call $\mathcal{P}(\mathcal{D}) = (\Pi(\mathcal{D}), \preceq)$ the *poset derived from distributive lattice* $\mathcal{D}$. We call $\mathcal{D}$ *simple* if the partition $\Pi(\mathcal{D})$ consists of singletons of V only. For a simple distributive lattice $\mathcal{D}$ we express $\mathcal{P}(\mathcal{D})$ by $(V, \preceq)$ instead of $(\{\{v\} \mid v \in V\}, \preceq)$.

A function $f : \mathcal{D} \to \mathbf{R}$ on a distributive lattice $\mathcal{D} \subseteq 2^V$ is called a *submodular function* if

$$\forall X, Y \in \mathcal{D} : f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y) \tag{1.32}$$

holds.

For a submodular function f on a distributive lattice $\mathcal{D} \subseteq 2^V$ with $\emptyset, V \in \mathcal{D}$ and $f(\emptyset) = 0$, we call the pair $(\mathcal{D}, f)$ a *submodular system* on V, and f the *rank function* of the submodular system. We call a submodular system $(\mathcal{D}, f)$ *simple* if $\mathcal{D}$ is simple. The polyhedron $\mathrm{P}(f)$ defined by

$$\mathrm{P}(f) = \{x \mid x \in \mathbf{R}^V, \forall X \in \mathcal{D}\colon x(X) \leq f(X)\} \tag{1.33}$$

is called the *submodular polyhedron* associated with the submodular system $(\mathcal{D}, f)$ on V. Also, the polyhedron $\mathrm{B}(f)$ defined by

$$\mathrm{B}(f) = \{x \mid x \in \mathrm{P}(f), x(V) = f(V)\} \tag{1.34}$$

is called the *base polyhedron* associated with the submodular system $(\mathcal{D}, f)$ on V.

A submodular system $(\mathcal{D}, f)$ on V is called *polymatroidal* if $\mathcal{D} = 2^V$ and f is monotone nondecreasing, i.e., for any $X \subseteq Y \subseteq V$ we have $f(X) \leq f(Y)$. A *polymatroid polyhedron* $\mathrm{P}_{(+)}(f)$ associated with polymatroidal $(2^V, f)$ is defined as

$$\mathrm{P}_{(+)}(f) = \{x \mid x \in \mathbf{R}^V_+, \forall X \subseteq V\colon x(X) \leq f(X)\}. \tag{1.35}$$

A polymatroidal submodular system $(2^V, f)$ is called *matroidal* if f is integer-valued and $f(X) \leq |X|$ for all $X \subseteq V$.

First, we give a fundamental but important theorem on base polyhedra.

Theorem 1.10 (Fujishige [36]): *The base polyhedron* $\mathrm{B}(f)$ *associated with the submodular system* $(\mathcal{D}, f)$ *on* V *is the set of all the maximal elements of* $\mathrm{P}(f)$. *In particular,* $\mathrm{B}(f) \neq \emptyset$. *Here, the partial order* $\preceq$ *among vectors in* $\mathbf{R}^V$ *is the one defined for vector lattice* $\mathbf{R}^V$ *(i.e., for* $x, y \in \mathbf{R}^V$ *we have* $x \preceq y$ *if and only if* $x(v) \leq y(v)$ *for all* $v \in V$*).* □

An example of submodular systems is given as follows (see [36]).

Proposition 1.11: *For a directed network* $\mathcal{N} = (G = (V, A_0), c)$*, define* $\mathcal{D} \subseteq 2^V$ *by* $\mathcal{D} = \{X \mid X \subseteq V, \kappa_c(X) < +\infty\}$*, where* κ_c *is the cut function of* $\mathcal{N}$ *(see* (1.13)*). Then,* $(\mathcal{D}, \kappa_c)$ *is a submodular system on* V*. Moreover, we have* $\partial\Phi = \mathrm{B}(\kappa_c)$*.* □

In the dual manner of defining submodular systems, we define supermodular systems as follows.

A function $g : \mathcal{D} \to \mathbf{R}$ on a distributive lattice $\mathcal{D} \subseteq 2^V$ is called a *supermodular function* if for each $X, Y \in \mathcal{D}$ we have

$$g(X) + g(Y) \leq g(X \cup Y) + g(X \cap Y). \tag{1.36}$$

For a supermodular function g on a distributive lattice $\mathcal{D} \subseteq 2^V$ with $\emptyset, V \in \mathcal{D}$ and $g(\emptyset) = 0$, we call the pair $(\mathcal{D}, g)$ a *supermodular system* on V, and g the *rank function* of the supermodular system. The polyhedron $\mathrm{P}(g)$ defined by

$$\mathrm{P}(g) = \{x \mid x \in \mathbf{R}^V, \forall X \in \mathcal{D}\colon x(X) \geq g(X)\} \tag{1.37}$$

is called the *supermodular polyhedron* associated with the supermodular system $(\mathcal{D}, g)$ on V. Also, the polyhedron $\mathrm{B}(g)$ defined by

$$\mathrm{B}(g) = \{x \mid x \in \mathrm{P}(g), x(V) = g(V)\} \tag{1.38}$$

is called the *base polyhedron* associated with the supermodular system $(\mathcal{D}, g)$ on V.

For a submodular system $(\mathcal{D}_1, f)$ on $V_1 \subseteq V$ and a supermodular system $(\mathcal{D}_2, g)$ on $V_2 \subseteq V$ such that

$$\begin{aligned} \forall X \in \mathcal{D}_1, Y \in \mathcal{D}_2: \;\; & X - Y \in \mathcal{D}_1, Y - X \in \mathcal{D}_2, \\ & f(X) - g(Y) \geq f(X - Y) - g(Y - X), \end{aligned} \tag{1.39}$$

the polyhedron $\mathrm{P}(f, g)$ defined by

$$\mathrm{P}(f, g) = \{x \mid x \in \mathbf{R}^V,\ \forall X \in \mathcal{D}_1\colon x(X) \leq f(X),\ \forall Y \in \mathcal{D}_2\colon x(Y) \geq g(Y)\} \tag{1.40}$$

is called a *generalized polymatroid* ([32]).

Chapter 2

Definitions and Preliminaries

We give definitions concerned with $\{\sqcup, \sqcap\}$-closed families, bisubmodular systems and bisubmodular polyhedra, and bidirected graphs. Also, we describe the preliminary results and facts that are needed in the later developments.

2.1. $\{\sqcup, \sqcap\}$-closed Families

Denote by 3^V the set of all the ordered pairs of disjoint subsets of V, i.e.,

$$3^V = \{(X,Y) \mid X, Y \subseteq V, X \cap Y = \emptyset\}. \tag{2.1}$$

For any $(X_i, Y_i) \in 3^V$ $(i = 1, 2)$ we write $(X_1, Y_1) \sqsubseteq (X_2, Y_2)$ if $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$. The binary relation $\sqsubseteq$ on 3^V is a partial order. We also write $(X_1, Y_1) \sqsubset (X_2, Y_2)$ if $(X_1, Y_1) \sqsubseteq (X_2, Y_2)$ and $(X_1, Y_1) \neq (X_2, Y_2)$. See Figure 2.1 for the case $V = \{1, 2\}$.

Each element (X, Y) of 3^V can be identified with the $\{0, \pm 1\}$-vector $\chi_{(X,Y)}$ defined by

$$\chi_{(X,Y)}(v) = \begin{cases} 1 & (v \in X) \\ -1 & (v \in Y) \\ 0 & (\text{otherwise}) \end{cases} \qquad (v \in V). \tag{2.2}$$

Because of this we call each element of 3^V a *signed subset* of V. Also, we call $\chi_{(X,Y)}$ the *characteristic vector* of signed subset (X, Y). We call $(\emptyset, \emptyset)$ the *null* signed subset of V.

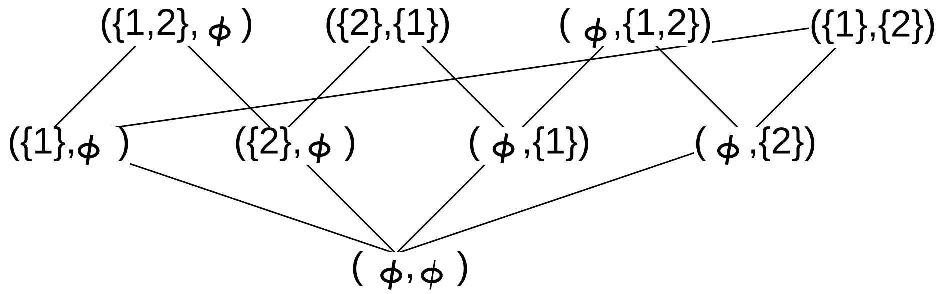

Figure 2.1: $3^{\{1,2\}}$ ordered by $\sqsubseteq$.

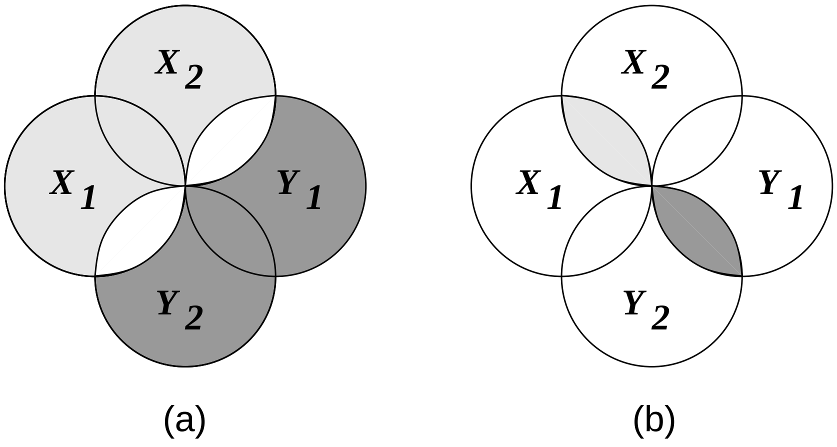

Figure 2.2: The Venn diagrams of $(X_1, Y_1) \sqcup (X_2, Y_2)$ and $(X_1, Y_1) \sqcap (X_2, Y_2)$.

For any signed subset $Z = (X, Y) \in 3^V$ we define

$$Z^+ = X, \quad Z^- = Y. \tag{2.3}$$

We call $Z^+(= X)$ $(Z^-(= Y))$ the *positive* (*negative*) *part* of the signed subset $Z = (X, Y)$.

We have two binary operations, *reduced union* $\sqcup$ and *intersection* $\sqcap$, on 3^V defined as

$$(X_1, Y_1) \sqcup (X_2, Y_2) = ((X_1 \cup X_2) - (Y_1 \cup Y_2), (Y_1 \cup Y_2) - (X_1 \cup X_2)), \tag{2.4}$$
$$(X_1, Y_1) \sqcap (X_2, Y_2) = (X_1 \cap X_2, Y_1 \cap Y_2) \tag{2.5}$$

for each $(X_1, Y_1), (X_2, Y_2) \in 3^V$. We call $\mathcal{F} \subseteq 3^V$ a $\{\sqcup, \sqcap\}$-*closed family* if it is closed under operations $\sqcup$ and $\sqcap$. For any $\mathcal{F} \subseteq 3^V$ we also call $\mathcal{F}$ a *family* on V.

Lemma 2.1: *For any $\{\sqcup, \sqcap\}$-closed family $\mathcal{F}$ every maximal element $(X, Y) \in \mathcal{F}$ has the same set $X \cup Y$, where the order among $\mathcal{F}$ is with respect to $\sqsubseteq$.*
(Proof) Since

$$(X_1 \cup (X_2 - Y_1), Y_1 \cup (Y_2 - X_1)) = ((X_1, Y_1) \sqcup (X_2, Y_2)) \sqcup (X_1, Y_1), \tag{2.6}$$

we have

$$(X_1 \cup (X_2 - Y_1), Y_1 \cup (Y_2 - Y_1)) \in \mathcal{F} \tag{2.7}$$

for any $(X_i, Y_i) \in \mathcal{F}$ $(i = 1, 2)$. Also, note that we have

$$(X_1, Y_1) \sqsubseteq (X_1 \cup (X_2 - Y_1), Y_1 \cup (Y_2 - X_1)), \tag{2.8}$$

$$X_1 \cup (X_2 - Y_1) \cup Y_1 \cup (Y_2 - X_1) = X_1 \cup X_2 \cup Y_1 \cup Y_2. \tag{2.9}$$

In particular, if (X_1, Y_1) and (X_2, Y_2) are two maximal elements in $\mathcal{F}$, we must have $X_1 \cup Y_1 = X_2 \cup Y_2$ by (2.8) and (2.9). □

Due to Lemma 2.1, for any maximal $(X,Y) \in \mathcal{F}$ let us call $X \cup Y$ the *support* of $\mathcal{F}$ and denote it by $\mathrm{Supp}(\mathcal{F})$.

If we have $\mathrm{Supp}(\mathcal{F}) = V$, we say $\mathcal{F}$ is a *spanning* family on V and $\mathcal{F}$ *spans* V. If $|\mathrm{Supp}(\mathcal{F})| = |V| - 1$, we call $\mathcal{F}$ *pre-spanning*. A $\{\sqcup, \sqcap\}$-closed family $\mathcal{F} \subseteq 3^V$ with $(\emptyset, \emptyset) \in \mathcal{F}$ is called *simple* if for each distinct $v, w \in \mathrm{Supp}(\mathcal{F})$ there exists an element $(X,Y) \in \mathcal{F}$ that separates v and w, i.e., $v \in X \cup Y$ and $w \notin X \cup Y$, or $v \notin X \cup Y$ and $w \in X \cup Y$. We call $\mathcal{F}$ with $(\emptyset, \emptyset) \in \mathcal{F}$ *pre-simple* if each distinct two vertices in $\mathrm{Supp}(\mathcal{F})$ except for one fixed pair of vertices are separated by an element of $\mathcal{F}$.

Let $\mathcal{F}$ be a $\{\sqcup, \sqcap\}$-closed family on V. A sequence of signed sets $(U_i, W_i) \in \mathcal{F}$ $(i = 0, 1, \cdots, k)$ is called a *chain* of $\mathcal{F}$ with respect to inclusion relation $\sqsubseteq$ if it satisfies

$$(U_0, W_0) \sqsubset (U_1, W_1) \sqsubset \cdots \sqsubset (U_k, W_k). \tag{2.10}$$

Here, k is the *length* of the chain. A chain $\mathcal{C}$ of $\mathcal{F}$ is called a *maximal chain* of $\mathcal{F}$ if there is no chain $\mathcal{C}'$ of $\mathcal{F}$ such that $\mathcal{C}'$ contains $\mathcal{C}$ as a proper subsequence. Every maximal chain of $\mathcal{F}$ has the same length (also see Lemma 4.11). If $\mathcal{F}$ is simple and spanning, then the length of any maximal chain of $\mathcal{F}$ is equal to $|V|$.

For any family $\mathcal{F} \subseteq 3^V$ and any subset $U \subseteq V$ the *reflection* $\mathcal{F} : U$ of $\mathcal{F}$ by U is define as

$$\mathcal{F} : U = \{(X,Y){:}U \mid (X,Y) \in \mathcal{F}\}, \tag{2.11}$$

where $(X,Y){:}U$ is defined by

$$(X,Y) : U = ((X - U) \cup (Y \cap U), (Y - U) \cup (X \cap U)) \tag{2.12}$$

for each $(X,Y) \in \mathcal{F}$ (see [19]). Note that if $\mathcal{F}$ is $\{\sqcup, \sqcap\}$-closed, then any reflection of $\mathcal{F}$ is again $\{\sqcup, \sqcap\}$-closed.

2.2. Bisubmodular Systems and Bisubmodular Polyhedra

A function $f \colon \mathcal{F} \to \mathbf{R}$ on a $\{\sqcup, \sqcap\}$-closed family $\mathcal{F} \subseteq 3^V$ is said to be *bisubmodular* if f satisfies

$$f(X_1, Y_1) + f(X_2, Y_2) \geq f((X_1, Y_1) \sqcup (X_2, Y_2)) + f((X_1, Y_1) \sqcap (X_2, Y_2)) \tag{2.13}$$

for any $(X_1, Y_1), (X_2, Y_2) \in \mathcal{F}$.

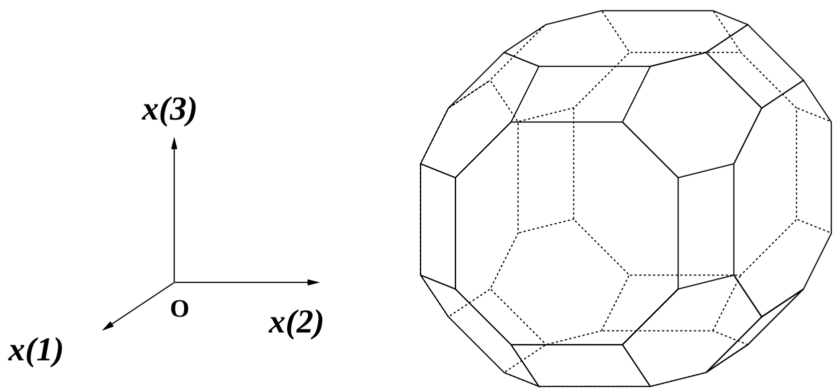

Figure 2.3: An example of a bisubmodular polyhedron.

Lemma 2.2: *Suppose that $f : \mathcal{F} \to \mathbf{R}$ is a bisubmodular function on a $\{\sqcup, \sqcap\}$-closed family $\mathcal{F}$. The set $\mathcal{F}_0$ of minimizers of f defined by*

$$\mathcal{F}_0 = \{(Z, W) \mid (Z, W) \in \mathcal{F}, f(Z, W) = \min\{f(X, Y) \mid (X, Y) \in \mathcal{F}\}\} \tag{2.14}$$

is $\{\sqcup, \sqcap\}$-closed. □

A function $f : \mathcal{F} \to \mathbf{R}$ on a $\{\sqcup, \sqcap\}$-closed family is called *bimodular* if (2.13) holds with equality for each $(X_i, Y_i) \in \mathcal{F}$ $(i = 1, 2)$. Note that any $x \in \mathbf{R}^V$ is a bimodular function.

A pair $(\mathcal{F}, f)$ of a spanning $\{\sqcup, \sqcap\}$-closed family $\mathcal{F} \subseteq 3^V$ on V with $(\emptyset, \emptyset) \in \mathcal{F}$ and a bisubmodular function $f : \mathcal{F} \to \mathbf{R}$ with $f(\emptyset, \emptyset) = 0$ is called a *bisubmodular system* on V. The *bisubmodular polyhedron* $\mathrm{P}_*(f)$ associated with the bisubmodular system $(\mathcal{F}, f)$ on V is defined by

$$\mathrm{P}_*(f) = \{x \mid x \in \mathbf{R}^V, \forall (X, Y) \in \mathcal{F} \colon x(X, Y) \leq f(X, Y)\}, \tag{2.15}$$

where for any $X \subseteq V$ $x(X) = \sum_{v \in X} x(v)$, $x(\emptyset) = 0$, and for any $(X, Y) \in 3^V$

$$x(X, Y) = x(X) - x(Y). \tag{2.16}$$

For a bisubmodular system $(\mathcal{F}, f)$ on V let (S, T) be any maximal element of $\mathcal{F}$. Note that $S \cup T = V$ since $\mathcal{F}$ spans V. We call such a pair (S, T) an *orthant.* For each orthant $(S, T) \in \mathcal{F}$ define $\mathcal{F}_{(S,T)} \subseteq \mathcal{F}$ by

$$\mathcal{F}_{(S,T)} = \{(X, Y) \mid (X, Y) \in \mathcal{F}, (X, Y) \sqsubseteq (S, T)\}. \tag{2.17}$$

Note that $\mathcal{F}_{(S,T)}$ forms a distributive lattice under $\sqcup$ and $\sqcap$ with the unique maximal element (S,T) and the unique minimal element $(\emptyset,\emptyset)$. We observe that

$$\mathcal{F} = \bigcup\{\mathcal{F}_{(S,T)} \mid (S,T) \text{ is an orthant in } \mathcal{F}\} \tag{2.18}$$

as well. Also, define $\mathcal{D}_{(S,T)} \subseteq 2^V$ by

$$\mathcal{D}_{(S,T)} = \{X \cup Y \mid (X,Y) \in \mathcal{F}_{(S,T)}\}. \tag{2.19}$$

It should be noted that $\mathcal{D}_{(S,T)}$ is a distributive lattice under lattice operations $\cup$ and $\cap$ with $\emptyset, E \in \mathcal{D}_Z$. For each orthant $(S,T) \in \mathcal{F}$, define a function $f_{(S,T)}\colon \mathcal{D}_{(S,T)} \to \mathbf{R}$ by

$$f_{(S,T)}(X) = f(X \cap S, X \cap T) \quad (X \in \mathcal{D}_{(S,T)}). \tag{2.20}$$

Define

$$\mathrm{P}_{(S,T)}(f) = \{x \mid x \in \mathbf{R}^V, \forall (X,Y) \in \mathcal{F}_{(S,T)}\colon x(X,Y) \leq f(X,Y)\}. \tag{2.21}$$

Then, from (2.18) we have

$$\mathrm{P}_*(f) = \bigcap\{\mathrm{P}_{(S,T)}(f) \mid (S,T) \text{ is an orthant in } \mathcal{F}\}. \tag{2.22}$$

Also, define

$$\mathrm{B}_{(S,T)}(f) = \{x \mid x \in \mathrm{P}_{(S,T)}(f), x(S,T) = f(S,T)\}. \tag{2.23}$$

We call $\mathrm{B}_{(S,T)}(f)$ the the *base polyhedron* of $(\mathcal{F}, f)$ *in the orthant* (S,T) of the bisubmodular polyhedron $\mathrm{P}_*(f)$.

For any subset $Q \subseteq \mathbf{R}^V$ and any subset U of V the *reflection* $Q\!:\!U$ of Q by U is defined as

$$Q\!:\!U = \{x\!:\!U \mid x \in Q\}, \tag{2.24}$$

where $x\!:\!U$ is defined by

$$(x\!:\!U)(v) = \begin{cases} -x(v) & (\text{if } v \in U) \\ x(v) & (\text{otherwise}) \end{cases} \qquad (v \in V). \tag{2.25}$$

The polyhedra $\mathrm{P}_{(S,T)}(f)$ and $\mathrm{B}_{(S,T)}(f)$ can be expressed by the submodular polyhedron and the base polyhedron associated with the submodular system $(\mathcal{D}_{(S,T)}, f_{(S,T)})$ on V as follows.

$$\mathrm{P}_{(S,T)}(f) = \mathrm{P}(f_{(S,T)})\!:\!T, \tag{2.26}$$
$$\mathrm{B}_{(S,T)}(f) = \mathrm{B}(f_{(S,T)})\!:\!T. \tag{2.27}$$

Therefore, the combinatorial properties of $\mathrm{P}_{(S,T)}(f)$ and $\mathrm{B}_{(S,T)}(f)$ are the same as those of $\mathrm{P}(f_{(S,T)})$ and $\mathrm{B}(f_{(S,T)})$, respectively. We see from Theorem 1.10 and (2.27) that $\mathrm{B}_{(S,T)}(f)$ is always nonempty.

The following lemma is fundamental.

Lemma 2.3: *Suppose that* $(\mathcal{F}, f)$ *is a bisubmodular system on* V. *The base polyhedron* $\mathrm{B}_{(S,T)}(f)$ *in each orthant* $(S,T) \in \mathcal{F}$ *of* $\mathrm{P}_*(f)$ *is contained in* $\mathrm{P}_*(f)$. *That is,*

$$\mathrm{B}_{(S,T)}(f) \subseteq \mathrm{P}_*(f) \quad ((S,T) \textit{ is an orthant in } \mathcal{F}). \tag{2.28}$$

(Proof) Suppose $x \in \mathrm{B}_{(S,T)}(f)$. Then for any $(X,Y) \in \mathcal{F}$, we have from (2.13)

$$\begin{aligned} x(X,Y) - f(X,Y) &= x(X,Y) - f(X,Y) + x(S,T) - f(S,T) \\ &\le x((X,Y) \sqcup (S,T)) + x((X,Y) \sqcap (S,T)) \\ &\quad - f((X,Y) \sqcup (S,T)) - f((X,Y) \sqcap (S,T)) \\ &\le 0. \end{aligned} \tag{2.29}$$

Hence, $x \in \mathrm{P}_*(f)$. □

We see from Lemma 2.3 that $\mathrm{B}_{(S,T)}(f)$ for each orthant $(S,T) \in \mathcal{F}$ is a face of $\mathrm{P}_*(f)$. Moreover, since $\mathrm{B}_{(S,T)}(f)$ is nonempty for each orthant $(S,T) \in \mathcal{F}$, $\mathrm{P}_*(f)$ is also nonempty.

For a bisubmodular system $(\mathcal{F}, f)$ on V define

$$\mathcal{F}^\diamond = \{(Y,X) \mid (X,Y) \in \mathcal{F}\}, \tag{2.30}$$

$$f^\diamond(Y,X) = f(X,Y) \quad ((X,Y) \in \mathcal{F}). \tag{2.31}$$

We call $(\mathcal{F}^\diamond, f^\diamond)$ the *dual bisubmodular system* of $(\mathcal{F}, f)$ on V. It follows that

$$\mathrm{P}_*(f) = -\mathrm{P}_*(f^\diamond). \tag{2.32}$$

A function $g\colon \mathcal{F} \to \mathbf{R}$ is called *bisupermodular* if $-g$ is bisubmodular. Also, $(\mathcal{F}, g)$ is called a *bisupermodular system* on V if $(\mathcal{F}, -g)$ is a bisubmodular system on V. The *bisupermodular polyhedron* $\mathrm{P}_*(g)$ associated with a bisupermodular system $(\mathcal{F}, g)$ on V is defined by

$$\mathrm{P}_*(g) = \{x \mid x \in \mathbf{R}^V, \forall (X,Y) \in \mathcal{F}\colon x(X,Y) \ge g(X,Y)\}. \tag{2.33}$$

For a bisubmodular system $(\mathcal{F}, f)$ we can easily see that

$$\mathrm{P}_*(-f) = -\mathrm{P}_*(f), \quad \mathrm{P}_*(f) = \mathrm{P}_*(-f^\diamond), \quad \mathrm{P}_*(-f) = \mathrm{P}_*(f^\diamond). \tag{2.34}$$

For a bisubmodular system $(\mathcal{F}, f)$ on V and a subset U of V we define the *reflection* of $(\mathcal{F}, f)$ by U as the bisubmodular system $(\mathcal{F}\!:\!U, f\!:\!U)$ on V, where

$$(f\!:\!U)((X,Y)\!:\!U) = f(X,Y) \qquad ((X,Y) \in \mathcal{F}). \tag{2.35}$$

We denote the reflection $(\mathcal{F}:U, f:U)$ by $(\mathcal{F}, f):U$. We can easily see that for any $U \subseteq V$

$$\mathrm{P}_*(f){:}U = \mathrm{P}_*(f{:}U) \tag{2.36}$$

(also see [19]).

2.3. Bidirected Graphs

Let $G = (V, A; \partial)$ be a bidirected graph. We recall that for each arc $a \in A$ there exist $v, w \in V$ such that one of the following three holds:

(1) $\partial a = v + w$ (arc a has two tails at v and w),

(2) $\partial a = -v - w$ (arc a has two heads at v and w),

(3) $\partial a = v - w$ (arc a has a tail at v and a head at w).

We call an arc a *type (1)* (respectively, *type (2)* and *type (3)*) if (1) (respectively, (2) and (3)) in the above holds.

We do not distinguish two bidirected graphs $G_1 = (V, A_1; \partial_1)$ and $G_2 = (V, A_2; \partial_2)$ such that $\{\partial_1 a \mid a \in A_1\} = \{\partial_2 a \mid a \in A_2\}$ holds.

When $\partial a = \pm v \pm w$, we say a is *incident to* v (and w), and if the coefficient of v in ∂a is positive (or negative), we say a is *positively* (or *negatively*) *incident to* v. If two arcs a and a' are, respectively, positively and negatively incident to a common vertex v, we say a and a' (or a' and a) are *oppositely incident to* v.

A *path* in a bidirected graph $G = (V, A; \partial)$ is an alternate sequence $P = (v_0, a_1, v_1, a_2, \cdots, a_k, v_k)$ of vertices v_i $(i = 0, 1, \cdots, k)$ and arcs a_i $(i = 1, 2, \cdots, k)$ for an integer $k \geq 1$ such that for each $i \in \{1, 2, \cdots, k\}$ arc a_i is incident to v_{i-1} and v_i. Vertex v_0 (v_k) is the *initial* (*terminal*) *vertex* of the path P and we say the path P is from v_0 to v_k. If we have $v_0 = v_k$, we say P is *closed.* Here, note that we allow repetitions of arcs in a path. If all the vertices $v_0, v_1, \cdots, v_k$ are distinct, the path P is called *simple*. A closed path $P = (v_0, a_1, v_1, a_2, \cdots, a_k, v_k)$ is called a *circle* if vertices $v_0, v_1, \cdots, v_{k-1}$ are distinct.

A bidirected graph $G = (V, A; \partial)$ is called *connected* if for any distinct two vertex $v, w \in V$ there is a path in G from v to w. A maximal connected subgraph of G is called a *connected component*.

Balancedness

A circle C is called *balanced* if the number of type (1) arcs in C plus the number of type (2) arcs in C is even and is *unbalanced* if it is not balanced. A bidirected graph is called *balanced* if it contains no unbalanced circle and *unbalanced* if it is not balanced. We call a balanced connected component a *balanced component*. For any subset U of vertex set V the *reflection* $G = (V, A; \partial)$ *by* U is the bidirected graph $G' = (V, A; \partial')$ defined as

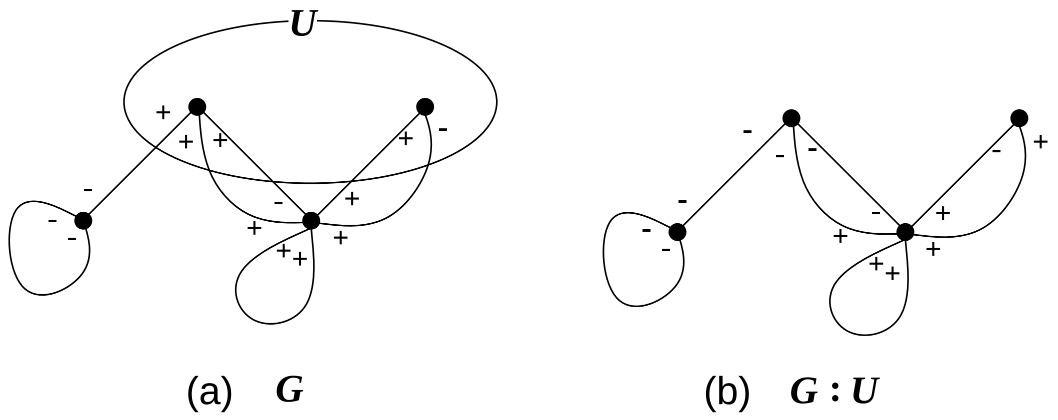

Figure 2.4: A reflection of a bidirected graph.

follows. For each arc $a \in A$, if $\partial a = \pm v \pm w$, we define

$$\partial' a = \pm\epsilon(v)v \pm \epsilon(w)w, \tag{2.37}$$

where for each $v \in V$

$$\epsilon(v) = \begin{cases} -1 & \text{if } v \in U \\ 1 & \text{if } v \notin U. \end{cases} \tag{2.38}$$

We denote the reflection G' by $G\!:\!U$. See Figure 2.4.

Lemma 2.4 (Harary [43]): *A bidirected graph $G = (V, A; \partial)$ is balanced if and only if for some $U \subseteq V$ the reflection $G\!:\!U$ of G by U is an ordinary directed graph, i.e., all the arcs of $G\!:\!U$ are of type (3) (one head and one tail).* □

A path $P = (v_0, a_1, v_1, a_2, \cdots, a_k, v_k)$ is *directed* if for each $i \in \{1, 2, \cdots, k-1\}$ arcs a_i and a_{i+1} are oppositely incident to vertex v_i. We define

$$\partial P = \sum_{i=1}^{k} \partial a_i. \tag{2.39}$$

A closed directed path $P = (v_0, a_1, \cdots, a_k, v_k)$ is called a *cycle* if $\partial P = 0$. A bidirected graph is called *acyclic* if it contains no cycle.

Circuits

Recall that for any function $\varphi\colon A \to \mathbf{R}$ the *boundary* $\partial\varphi\colon V \to \mathbf{R}$ of φ is given by

$$\partial\varphi = \sum\{\varphi(a)\partial a \mid a \in A\}. \tag{2.40}$$

For a bidirected graph $G = (V, A; \partial)$ a function $\varphi\colon A \to \mathbf{R}$ is called a *circulation* in G if $\partial\varphi = \mathbf{0}$. For a circulation φ in G the *support* $\mathrm{Supp}(\varphi)$ of φ is defined as

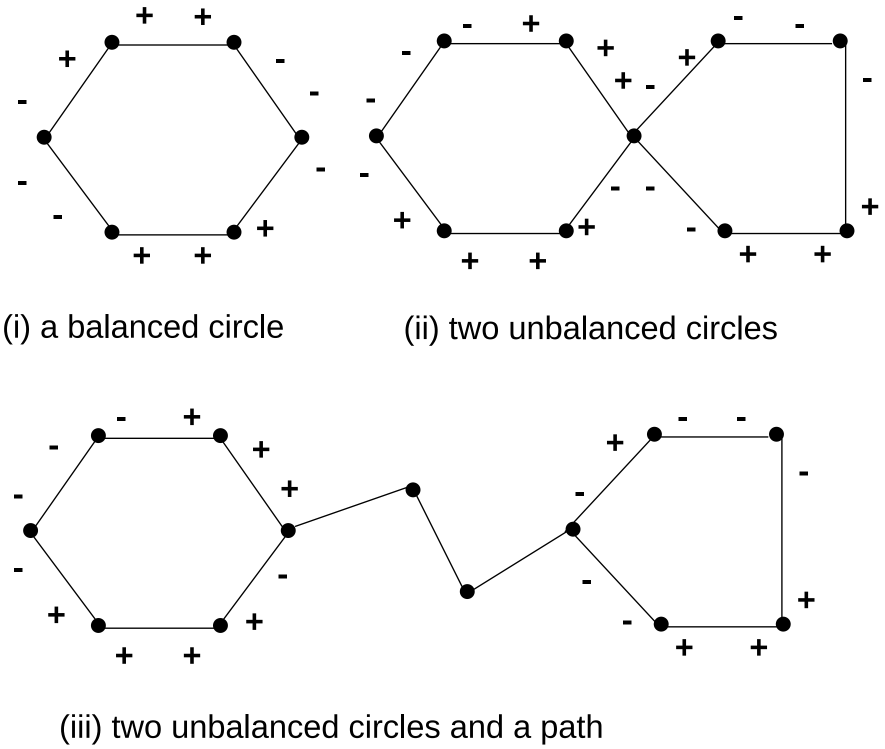

Figure 2.5: The three types of circuits in a bidirected graph.

$\mathrm{Supp}(\varphi) = \{a \mid a \in A, \varphi(a) \neq 0\}$. A circulation φ in G is called *elementary* if there is no circulation φ' in G such that $\emptyset \neq \mathrm{Supp}(\varphi') \subset \mathrm{Supp}(\varphi)$ (strict inclusion). The support of an elementary circulation in a bidirected graph G is called a *circuit* of G. Circuits of bidirected graphs are characterized as follows.

Theorem 2.5 (Zaslavsky [85] (see also [16])): *For any bidirected graph a subset of arcs is a circuit if and only if it is either*

(i) *a balanced circle,*

(ii) *the union of two unbalanced circles which have exactly one vertex in common, or*

(iii) *the union of two vertex disjoint unbalanced circles C_1 and C_2 and a simple path meeting each of the two circles at exactly one of its end points.* □

For any circuit C one can easily see that there exists a $\{0, \pm 1, \pm 2\}$-valued vector $\chi_C : A \to \mathbf{R}$ such that $\sum_{a \in A} \chi_C(a)\partial a = 0$, where if C is of type (i), then we can choose χ_C to be $\{0, \pm 1\}$-valued. Such χ_C is determined uniquely up to multiple by -1. We call χ_C the *characteristic vector* of circuit C. Each elementary circulation is a scalar multiple of the characteristic vector of a circuit.

For two circulations φ_1 and φ_2 in a bidirected graph $G = (V, A; \partial)$ we say φ_1 *conforms* to φ_2 if $\varphi_1(a) \neq 0$ implies $\varphi_1(a)\varphi_2(a) > 0$ for each $a \in A$.

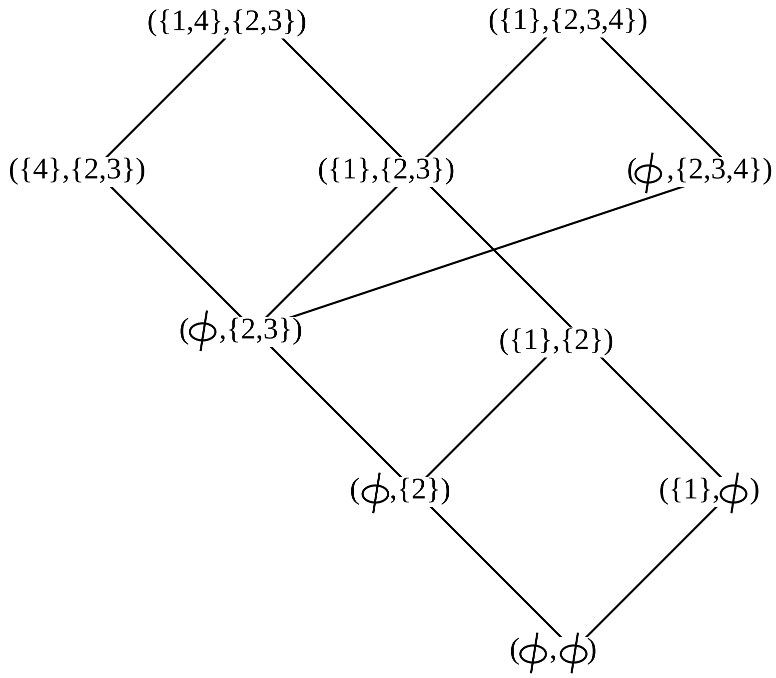

Figure 2.6: The set of ideals of the bidirected graph in Figure 1.2 ordered by $\sqsubseteq$.

Specializing a result of Rockafellar [74] we have the following lemma (see also [39]).

Lemma 2.6: *For any circulation φ in a bidirected graph G there exist circuits $C_1, \cdots, C_k$ of G and $\lambda_i > 0$ $(i = 1, \cdots, k)$ such that for each $i = 1, \cdots, k$ χ_{C^i} conforms to φ and $\varphi = \sum_{i=1}^{k} \lambda_i \chi_{C^i}$.* □

Ideals

For a bidirected graph $G = (V, A; \partial)$ a signed subset $(X, Y) \in 3^V$ is an *ideal* of G if it satisfies

$$\langle \partial a, (X, Y)\rangle \ \leq \ 0 \quad (a \in A), \tag{2.41}$$

where ∂a and (X, Y) should be regarded as integral vectors in $\mathbf{R}^V$ in a natural way, and $\langle \cdot, \cdot \rangle$ is the canonical inner product. For each $a \in A$ we denote by $\partial^+ a$ $(\partial^- a)$ the set of the vertices that have positive (negative) coefficients in ∂a. Inequality (2.41) means that $(\partial^+ a \cap X) \cup (\partial^- a \cap Y) \neq \emptyset$ implies $(\partial^- a \cap X) \cup (\partial^+ a \cap Y) \neq \emptyset$. In Reiner's definition of ideal the inequality sign in (2.41) is reversed but we adopt the above definition due only to the consistency with our system of notations for ordinary posets and ideals (cf. [36]).

One can easily verify the following.

Lemma 2.7: *Suppose that $G = (V, A; \partial)$ is a bidirected graph. For any subset $U \subseteq V$ we have (X, Y) is an ideal of G if and only if $(X, Y){:}U$ is an ideal of $G{:}U$.* □

We have an another characterization of balanced bidirected graphs in terms of their ideals.

Lemma 2.8: *A bidirected graph $G = (V, A; \partial)$ is balanced if and only if there exists an orthant $(S, T) \in 3^V$ such that both of (S, T) and (T, S) are ideals of G.*
(Proof) Suppose that a bidirected graph $G = (V, A; \partial)$ is balanced. Then, by Lemma 2.4 there exists a subset U of V such that $G\!:\!U$ is an ordinary directed graph. Since both of $(V, \emptyset)$ and $(\emptyset, V)$ is ideals of $G\!:\!U$, it follows from Lemma 2.7 that both of $(V - U, U)$ and $(U, V - U)$ are ideals of G.

Conversely, if $(V - U, U)$ and $(U, V - U)$ are ideals of G for some $U \subseteq V$, then $(V, \emptyset)$ and $(\emptyset, V)$ are ideals of $G\!:\!U$. This implies there is no arc of type (1) or of type (2) in $G\!:\!U$. It follows from Lemma 2.4 that G is balanced. $\square$

Chapter 3

Combinatorial Optimization Problems on Bidirected Graphs and Networks

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As a generalization of undirected and directed graphs J. Edmonds and E. L. Johnson [30] considered bidirected graphs and they showed that many graph-theoretic problems on bidirected graphs can be well-solved. Later, Johnson and Padberg [50] considered bidirected graphs as representations of systems of degree-two inequalities. Very closely related to the concept of bidirected graph is that of signed graph, which was extensively studied by T. Zaslavsky ([85], [86]). Indeed, a bidirected graph can be considered as an orientation of a signed graph. Based on the works of Zaslavsky, Bouchet [16] considered nowhere-zero flows on bidirected graphs and Khelladi [53] followed him. Khelladi also studied two-connectivity of bidirected graphs. Bouchet and Cunningham [19] dealt with the degree sequences of bidirected graphs and showed the convex hull of the degree sequences of a bidirected graph is a bisubmodular polyhedron. The descriptions of the polyhedra were given by Ando, Fujishige and Naitoh [8] in terms of the cut functions of bidirected graphs. The strong connectivity of a bidirected graph is studied by Ando, Fujishige and Nemoto [10] and the signed poset structure of the strongly connected components is revealed. An interesting result concerned with the strongly connected component decompositions was obtained by Iwata [48] on the Dulmage-Mendelsohn type decompositions of general (undirected) graphs.

The importance of bidirected graphs and networks in studies of bisubmodular system is twofold. First, the cut function of any bidirected network is a bisubmodular function and the associated polyhedron is exactly the set of the boundaries of the flows in the bidirected network. Secondly, we associate a bidirected graph $G(x)$ called the exchangeability graph for each point x in a bisubmodular polyhedron $\mathrm{P}_*(f)$. The exchangeability graph represent the feasible direction at x in $\mathrm{P}_*(f)$ and contains useful information about the neighbor of x. Hence, the study of bidirected graphs is indispensable for investigations of bisubmodular systems and bisubmodular polyhedra.

A signed covering graph is an ordinary directed graph which represents the incidence relation of a bidirected graph. Signed covering graphs are called implication graphs in studies of quadratic Boolean equations (see, e.g., [12], [42]). Zaslavsky used signed covering graphs to investigate structures of matroids of signed (or bidirected) graphs where the concept of signed covering graph is attributed to Biggs [13]. S. D. Fischer [31] used signed covering graphs to analyze signed posets. Recently, Ando, Fujishige and Nemoto showed that both of the problem of finding the strongly connected component decompositions of bidirected graphs [10] and the minimum-weight ideal problem for bidirected graphs [11] are reduced to those problems on their signed covering graphs.

In this chapter the use of signed covering graph is fully exploited for solving some optimization problems on bidirected graphs. We show, in Section 3.2, that the set of boundaries of a bidirected graph forms a bisubmodular polyhedron and an interesting relationship with a base polyhedron is revealed. In Section 3.3 we consider the transitive closures and the Hasse diagrams of bidirected graphs. In Section 3.4 we consider the minimum cost circulation problem and show that it can be reduced to that on ordinary directed graphs. Also, the half-integrality of an optimal solution is easily derived. In Section 3.5 we deal with the minimum cut problem and the maximum flow problem and show that the maxflow-mincut theorem is generalized for bidirected networks. Finally, the minimum-weight ideal problem is considered in Section 3.6.

This chapter not only serves as the foundation for the developments in the later chapters but also contains some important results in its own rights.

3.1. Bidirected Graphs and Their Signed Covering Graphs

Given a bidirected graph $G = (V, A; \partial)$, the *signed covering graph* ([86], also see [13], [85]) $\tilde{G} = (\tilde{V}, \tilde{A}; \tilde{\partial})$ of G is an ordinary directed graph defined as follows. The vertex set $\tilde{V}$ is given by

$$\tilde{V} = V \times \{+, -\} \tag{3.1}$$

and the arc set $\tilde{A}$ by

$$\tilde{A} = \{a^{(+)} \mid a \in A\} \bigcup \{a^{(-)} \mid a \in A\}. \tag{3.2}$$

Moreover, the boundary operator $\tilde{\partial}$ in $\tilde{G}$ is defined as follows.
For each $a \in A$,

(i) if $\partial a = v + w$, then $\tilde{\partial} a^{(+)} = (v, +) - (w, -)$, $\quad \tilde{\partial} a^{(-)} = (w, +) - (v, -)$,
(ii) if $\partial a = -v - w$, then $\tilde{\partial} a^{(+)} = (v, -) - (w, +)$, $\quad \tilde{\partial} a^{(-)} = (w, -) - (v, +)$,
(iii) if $\partial a = v - w$, then $\tilde{\partial} a^{(+)} = (v, +) - (w, +)$, $\quad \tilde{\partial} a^{(-)} = (w, -) - (v, -)$,

where since $v + w = w + v$ and $-v - w = -w - v$, $a^{(+)}$ and $a^{(-)}$ in (i) (and (ii)) may be interchanged but this ambiguity is inessential. See Figure 3.1.

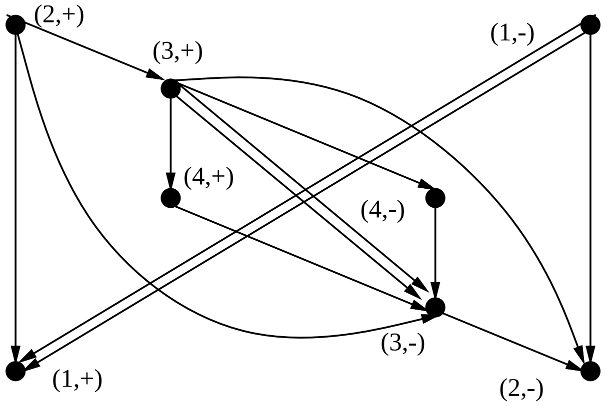

Figure 3.1: The signed covering graph of the bidirected graph in Figure 1.2.

By the definition, the signed covering graph satisfies the following condition:

(SD) If there exists an arc $\tilde{a} \in \tilde{A}$ such that $\tilde{\partial}\tilde{a} = (u, \sigma) - (v, \tau)$, then there exists an arc $\tilde{a}^* \in \tilde{A}$ such that $\tilde{\partial}\tilde{a}^* = (v, -\tau) - (u, -\sigma)$.

Conversely, for any directed graph $\tilde{G} = (\tilde{V}, \tilde{A}; \tilde{\partial})$ on $\tilde{V}$ satisfying Condition (SD) we can define a bidirected graph G in a reversed manner as above and we call such a directed graph $\tilde{G}$ *self-dual* ([31]).

Directed paths in G are related to directed paths in $\tilde{G}$ as follows.

Lemma 3.1: *For any vertices $u, w \in V$ and any signs $\sigma(u), \sigma(w) \in \{+, -\}$,*

$$P = (v_0(= u), a_1, v_1, \cdots, a_k, v_k(= w)) \tag{3.3}$$

is a directed path from u to w in G such that $\partial P = \sigma(u)u + \sigma(w)w$ if and only if both

$$\begin{aligned} \tilde{P}_1 &= ((v_0, \sigma_0), a_1^{(\tau_1)}, (v_1, \sigma_1), \cdots, (v_{k-1}, \sigma_{k-1}), a_k^{(\tau_k)}, (v_k, -\sigma_k)), && (3.4) \\ \tilde{P}_2 &= ((v_k, \sigma_k), a_k^{(-\tau_k)}, (v_{k-1}, -\sigma_{k-1}), \cdots, (v_1, -\sigma_1), a_1^{(-\tau_1)}, (v_0, -\sigma_0)) && (3.5) \end{aligned}$$

are directed paths in $\tilde{G}$ for some signs $\sigma_i, \tau_j \in \{+, -\}$ $(i = 1, 2, \cdots, k-1; j = 1, 2, \cdots, k)$, where $v_0 = u$, $v_k = w$, $\sigma_0 = \sigma(u)$ and $\sigma_k = \sigma(w)$.

(Proof) The present lemma easily follows from the definition of the signed covering graph $\tilde{G}$. □

The following lemma is fundamental in the subsequent discussions.

Lemma 3.2: *Suppose that $G = (V, A; \partial)$ is a bidirected graph and $\tilde{G} = (\tilde{V}, \tilde{A}; \tilde{\partial})$ is its signed covering graph. Let us denote the boundary operators of G and $\tilde{G}$ by ∂ and $\tilde{\partial}$,*

respectively.
(i) *For any function* $\tilde{\varphi}\colon \tilde{A} \to \mathbf{R}$*, if* $\varphi\colon A \to \mathbf{R}$ *is such that*

$$\varphi(a) = \frac{1}{2}(\tilde{\varphi}(a^{(+)}) + \tilde{\varphi}(a^{(-)})) \quad (a \in A), \tag{3.6}$$

then we have

$$\partial\varphi(v) = \frac{1}{2}(\tilde{\partial}\tilde{\varphi}(v,+) - \tilde{\partial}\tilde{\varphi}(v,-)) \quad (v \in V). \tag{3.7}$$

(ii) *Given any function* $\varphi\colon A \to \mathbf{R}$*, define* $\tilde{\varphi}\colon \tilde{A} \to \mathbf{R}$ *by*

$$\tilde{\varphi}(a^{(+)}) = \tilde{\varphi}(a^{(-)}) = \varphi(a) \quad (a \in A). \tag{3.8}$$

Then we have

$$\tilde{\partial}\tilde{\varphi}(v,\sigma) = \sigma\partial\varphi(v) \quad ((v,\sigma) \in \tilde{V}). \tag{3.9}$$

(Proof) (i) For each $v \in V$ we have

$$\begin{aligned}
&\tilde{\partial}\tilde{\varphi}(v,+) - \tilde{\partial}\tilde{\varphi}(v,-) \\
&= \sum\{\tilde{\varphi}(\tilde{a}) \mid \tilde{a} \in \delta^+(v,+)\} + \sum\{\tilde{\varphi}(\tilde{a}) \mid \tilde{a} \in \delta^-(v,-)\} \\
&\quad - \sum\{\tilde{\varphi}(\tilde{a}) \mid \tilde{a} \in \delta^-(v,+)\} - \sum\{\tilde{\varphi}(\tilde{a}) \mid \tilde{a} \in \delta^+(v,-)\} \\
&= \sum\{\tilde{\varphi}(a^{(+)}) + \tilde{\varphi}(a^{(-)}) \mid a \in A,\ a\text{: nonselfloop arc positively incident to } v\} \\
&\quad +2\sum\{\tilde{\varphi}(a^{(+)}) + \tilde{\varphi}(a^{(-)}) \mid a \in A,\ a\text{: selfloop positively incident to } v\} \\
&\quad - \sum\{\tilde{\varphi}(a^{(+)}) + \tilde{\varphi}(a^{(-)}) \mid a \in A,\ a\text{: nonselfloop arc negatively incident to } v\} \\
&\quad -2\sum\{\tilde{\varphi}(a^{(+)}) + \tilde{\varphi}(a^{(-)}) \mid a \in A,\ a\text{: selfloop negatively incident to } v\} \\
&= 2\sum\{\varphi(a) \mid a \in A,\ a\text{: nonselfloop arc positively incident to } v\} \\
&\quad +4\sum\{\varphi(a) \mid a \in A,\ a\text{: selfloop positively incident to } v\} \\
&\quad -2\sum\{\varphi(a) \mid a \in A,\ a\text{: nonselfloop arc negatively incident to } v\} \\
&\quad -4\sum\{\varphi(a) \mid a \in A,\ a\text{: selfloop negatively incident to } v\} \\
&= 2\partial\varphi(v).
\end{aligned} \tag{3.10}$$

(ii) The mappings

$$\delta^+(v,+) \ni a^{(\epsilon)} \ \mapsto \ a^{(-\epsilon)} \in \delta^-(v,-), \tag{3.11}$$

$$\delta^-(v,+) \ni a^{(\epsilon)} \ \mapsto \ a^{(-\epsilon)} \in \delta^+(v,-) \tag{3.12}$$

are bijections. Hence, together with the definition (3.8) we have, $\tilde{\partial}\tilde{\varphi}(v,+) = -\tilde{\partial}\tilde{\varphi}(v,-)$. Then, by (i) we have (3.9). □

3.2. Boundaries of Flows in Bidirected Networks

Let $\mathcal{N} = (G = (V, A), c)$ be a bidirected network, where we allow the capacity function c to take value $+\infty$. Recall that the cut function $\kappa_c\colon 3^V \to \mathbf{R} \cup \{+\infty\}$ of $\mathcal{N}$ is defined as

$$\kappa_c(X, Y) = \sum\{\langle \partial a, (X, Y)\rangle c(a) \mid a \in A, \langle \partial a, (X, Y)\rangle > 0\} \quad ((X, Y) \in 3^V). \tag{3.13}$$

The *signed covering network* $\tilde{\mathcal{N}} = (\tilde{G} = (\tilde{V}, \tilde{A}; \tilde{\partial}), \tilde{c})$ *of* $\mathcal{N}$ is the pair of the signed covering graph $\tilde{G}$ of G and $\tilde{c}\colon \tilde{A} \to \mathbf{R} \cup \{+\infty\}$ defined by $\tilde{c}(a^{(+)}) = \tilde{c}(a^{(-)}) = c(a)$ $(a \in A)$. In the following, we sometimes write c instead of $\tilde{c}$ for the simplicity of the notation. For the signed covering network $\tilde{\mathcal{N}}$ of $\mathcal{N}$ its cut function $\tilde{\kappa}_c\colon 2^{\tilde{V}} \to \mathbf{R} \cup \{+\infty\}$ is defined as

$$\tilde{\kappa}_c(\tilde{X}) = \sum\{c(\tilde{a}) \mid \tilde{a} \in \Delta^+(\tilde{X})\} \quad (\tilde{X} \in 2^{\tilde{V}}), \tag{3.14}$$

where $\Delta^+ X$ denotes the set of arcs leaving $\tilde{X}$ (see Section 1.3.b).

For any subset X of V denote, by $(X, +)$ and $(X, -)$, the subsets $\{(v, +) \mid v \in X\}$ and $\{(v, -) \mid v \in X\}$ of $\tilde{V}$, respectively. By an elementary counting, we have the following lemma.

Lemma 3.3: *For any* $(X, Y) \in 3^V$ *we have*

$$\kappa_c(X, Y) = \tilde{\kappa}_c((X, +) \cup (Y, -)). \tag{3.15}$$

□

For any subset $\tilde{X}$ of $\tilde{V}$ denote by $\tilde{X}^*$ the subset $\{(v, -\sigma) \mid (v, \sigma) \in X\}$ of $\tilde{V}$.

Lemma 3.4: *For any* $\tilde{X} \subseteq \tilde{V}$ *we have*

$$\tilde{\kappa}_c(\tilde{X}) \geq \tilde{\kappa}(\tilde{X} - \tilde{X}^*). \tag{3.16}$$

(Proof) For any $\tilde{X} \subseteq \tilde{V}$ we have

$$\tilde{\kappa}_c(\tilde{X}) = \tilde{\kappa}_c(\tilde{V} - \tilde{X}^*) \tag{3.17}$$

since the mapping $\Delta^+\tilde{X} \ni a^{(\epsilon)} \mapsto a^{(-\epsilon)} \in \Delta^-\tilde{X}^*$ is a bijection and $c(a^{(\epsilon)}) = c(a^{(-\epsilon)})$ by definition. Therefore, by the submodularity of $\tilde{\kappa}_c$ (see Proposition 1.11), we have

$$\begin{aligned} 2\tilde{\kappa}_c(\tilde{X}) &= \tilde{\kappa}_c(\tilde{X}) + \tilde{\kappa}_c(\tilde{V} - \tilde{X}^*) \\ &\geq \tilde{\kappa}_c(\tilde{V} - (\tilde{X}^* - \tilde{X})) + \tilde{\kappa}_c(\tilde{X} - \tilde{X}^*) \\ &= \tilde{\kappa}_c(\tilde{V} - (\tilde{X} - \tilde{X}^*)^*) + \tilde{\kappa}_c(\tilde{X} - \tilde{X}^*) \\ &= 2\tilde{\kappa}_c(\tilde{X} - \tilde{X}^*) \end{aligned} \tag{3.18}$$

for any $\tilde{X} \subseteq \tilde{V}$ and the present lemma holds. □

We see from Lemmas 3.3 and 3.4 that for any $(X_1, Y_1), (X_2, Y_2) \in 3^V$ we have

$$\begin{aligned}
&\kappa_c(X_1, Y_1) + \kappa_c(X_2, Y_2) \\
&\quad = \tilde{\kappa}_c((X_1, +) \cup (Y_1, -)) + \tilde{\kappa}_c((X_2, +) \cup (Y_2, -)) \\
&\quad \geq \tilde{\kappa}_c(((X_1, +) \cup (Y_1, -)) \cup ((X_2, +) \cup (Y_2, -))) \\
&\qquad + \tilde{\kappa}_c(((X_1, +) \cup (Y_1, -)) \cap ((X_2, +) \cup (Y_2, -))) \\
&\quad \geq \tilde{\kappa}_c(((X_1 \cup X_2) - (Y_1 \cup Y_2), +) \cup ((Y_1 \cup Y_2) - (X_1 \cup X_2), -))) \\
&\qquad + \tilde{\kappa}_c((X_1 \cap X_2, +) \cup (Y_1 \cap Y_2, -)) \\
&\quad = \kappa_c((X_1 \cup X_2) - (Y_1 \cup Y_2), (Y_1 \cup Y_2) - (X_1 \cup X_2)) \\
&\qquad + \kappa_c(X_1 \cap X_2, Y_1 \cap Y_2) \\
&\quad = \kappa_c((X_1, Y_1) \sqcup (X_2, Y_2)) + \kappa_c((X_1, Y_1) \sqcap (X_2, Y_2)).
\end{aligned} \tag{3.19}$$

Here, the first inequality is due to the submodularity of $\tilde{\kappa}_c$. Hence, $\kappa_c \colon 3^V \to \mathbf{R} \cup \{+\infty\}$ is a bisubmodular function.

Define

$$\mathcal{F} = \{(U, W) \mid (U, W) \in 3^V, \kappa_c(U, W) < +\infty\}. \tag{3.20}$$

Then, we have

Lemma 3.5: $(\mathcal{F}, \kappa_c)$ *is a bisubmodular system on* $\mathrm{Supp}(\mathcal{F})$. □

We see the following interesting relationship between a bisubmodular polyhedron and a base polyhedron.

Suppose that $\mathcal{F}$ define by (3.5) spans V, i.e., $\mathrm{Supp}(\mathcal{F}) = V$ and that $\mathrm{B}(\tilde{\kappa})$ is the base polyhedron associated with the submodular system $(\mathcal{D}, \tilde{\kappa_c})$ on $\tilde{V}$, where $\tilde{\kappa_c}$ is the cut function of the signed covering network $\tilde{\mathcal{N}}$ of $\mathcal{N}$ and $\mathcal{D}$ is the distributive lattice defined by

$$\mathcal{D} = \{\tilde{X} \mid \tilde{X} \subseteq \tilde{V}, \tilde{\kappa_c}(\tilde{X}) < +\infty\}. \tag{3.21}$$

Theorem 3.6: *We have*

$$\mathrm{P}_*(\kappa_c) = \{x \mid x \in \mathbf{R}^V, \tilde{x} \in \mathrm{B}(\tilde{\kappa}_c)\}, \tag{3.22}$$

where for $x \in \mathbf{R}^V$ $\tilde{x} \in \mathbf{R}^{\tilde{V}}$ *is defined by*

$$\tilde{x}(v, \sigma) = \sigma x(v) \quad ((v, \sigma) \in \tilde{V}). \tag{3.23}$$

(Proof) Since

$$x(X, Y) = x(X) - x(Y) = \tilde{x}(X, +) + \tilde{x}(Y, -) = \tilde{x}((X, +) \cup (Y, -)), \tag{3.24}$$

by Lemma 3.3 we have $x \in \mathrm{P}_*(\kappa_c)$ if and only if $\tilde{x}((X,+) \cup (Y,-)) \leq \tilde{\kappa}_c((X,+) \cup (Y,-))$ holds for any $(X,Y) \in \mathcal{F}$. By virtue of Lemma 3.4, this is equivalent to $\tilde{x}(\tilde{X}) \leq \tilde{\kappa}_c(\tilde{X})$ for any $\tilde{X} \in \mathcal{D}$, i.e., $\tilde{x} \in \mathrm{B}(\tilde{\kappa}_c)$. □

Let us denote the set of all the boundaries of bidirected flows in $\mathcal{N}$ by $\partial\Phi$. From Theorem 3.6 we have a characterization of the boundaries of flows in a bidirected network as follows.

Corollary 3.7: *We have*

$$\partial\Phi = \mathrm{P}_*(\kappa_c). \tag{3.25}$$

(Proof) Suppose $x \in \mathrm{P}_*(\kappa_c)$. By Theorem 3.6, $\tilde{x}$ defined by (3.23) is in the base polyhedron $\mathrm{B}(\tilde{\kappa}_c)$. It follows from Proposition 1.11 that there exist a flow $\tilde{\varphi}$ in the covering network $\tilde{\mathcal{N}}$ such that $\tilde{\partial}\tilde{\varphi} = \tilde{x}$. It follows from Lemma 3.2 that for the feasible flow φ in $\mathcal{N}$ defined by (3.8) we have

$$\partial\varphi(v) = x(v). \tag{3.26}$$

Hence, $x \in \partial\Phi$. The converse inclusion $\partial\Phi \subseteq \mathrm{P}_*(\kappa_c)$ is trivial. □

It should be noted that the degree sequence polyhedron of an undirected graph is a special case of (3.25), where $A^- = A^0 = \emptyset$ (see [22] and also [19] for the generalizations and related topics).

3.3. Transitive Closures and the Hasse Diagrams

For any bidirected graph $G = (V, A; \partial)$ the *transitive closure* of G, denoted by G^*, is the bidirected graph constructed from G by repeating the following operations (1) and (2) until no new arc can be generated:

(1) For any two arcs a_1, a_2 in G that are not both selfloops and are oppositely incident to exactly one common vertex, if there is no arc a_3 in G such that $\partial a_3 = \partial a_1 + \partial a_2$, then add such an arc a_3 to G.
(2) For any two selfloops a_1, a_2 in G incident to distinct vertices, if there is no arc a_3 in G such that $2\partial a_3 = \partial a_1 + \partial a_2$, then add such an arc a_3 to G.

See Figure 3.2.a.

For each $i = 1, 2$ let G^{*_i} be the bidirected graph obtained from a bidirected graph $G = (V, A; \partial)$ by repeating the operation (i) in the above definition of the transitive closure. We call G^{*_i} the (i)-*transitive closure* of G for $i = 1, 2$. See Figure 3.3.a.

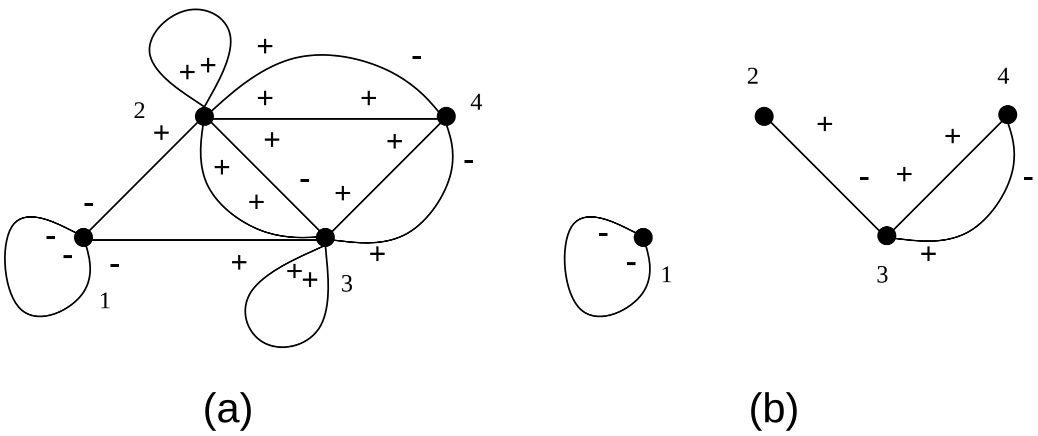

Figure 3.2: The transitive closure and the Hasse diagram of the bidirected graph in Figure 1.2.

Lemma 3.8: *For any bidirected graph we have*

$$G^* = (G^{*_1})^{*_2}. \tag{3.27}$$

(Proof) Since G is a subgraph of $(G^{*_1})^{*_2}$, we only have to show that $(G^{*_1})^{*_2}$ is transitively closed, i.e., we cannot apply either the operation (1) or the operation (2) to $(G^{*_1})^{*_2}$. Clearly, the applicability of the operation (2) is excluded. Let us consider the applicability of the operation (1).

Suppose that there exist arcs a_1 and a_2 of $(G^{*_1})^{*_2}$ oppositely incident to a common vertex $v \in V$ such that $\partial a_1 + \partial a_2 \neq 0$.

Case (i): Both a_1 and a_2 are arcs of G^{*_1}. By the definition of G^{*_1} there exists an arc a_3 of G^{*_1} such that $\partial a_3 = \partial a_1 + \partial a_2$.

Case (ii): Neither a_1 nor a_2 is an arc of G^{*_1}. Then, these two arcs are added by the operation (2), and hence, are not selfloop. Suppose that $\partial a_1 = \sigma u + v$ and $\partial a_2 = -v + \tau w$ for some $\sigma, \tau \in \{+, -\}$ and $u, v, w \in V$ without loss of generality. By the definition of the operation (2), there are selfloops rooted at u and w with their boundaries, respectively, $2\sigma u$ and $2\tau w$. If $u = w$, then $\sigma = \tau$ since $\partial a_1 + \partial a_2 \neq 0$, and the selfloop rooted at $u(= w)$ is the desired arc a_3. If $u \neq w$, then by the definition of the operation (2), there must be an arc a_3 such that $\partial a_3 = \partial a_1 + \partial a_2$.

Case (iii): a_1 is an arc of G^{*_1} but a_2 is not. Since a_2 is not a selfloop arc, let $\partial a_2 = \sigma v + \tau w$. By the definition of G^{*_2}, there exist selfloop arcs a_3 and a_4 of G^{*_1} such that $\partial a_3 = 2\sigma v$ and $\partial a_4 = 2\tau w$. If a_1 is a selfloop with its boundary $\partial a_1 = -2\sigma v$, then by the definition of the operation (2) there exists an arc a' such that $\partial a' = -\sigma v + \tau w$.

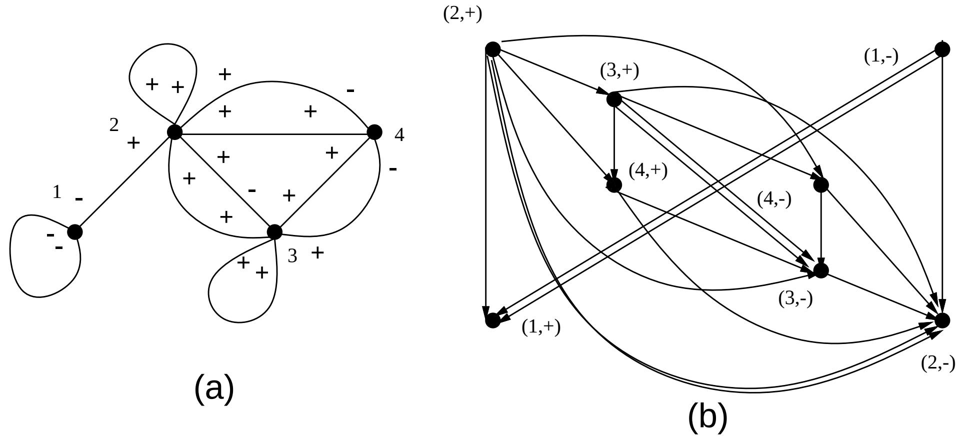

Figure 3.3: The (1)-transitive closure G^{*_1} and its signed covering graph $\widetilde{G^{*_1}}$ of the bidirected graph in Figure 1.2.

Hence, suppose a_1 is not a selfloop, and let $\partial a_1 = -\sigma v + \nu u$. If $u = w$, then $\nu = \tau$ by the assumption $\partial a_1 + \partial a_2 \neq 0$ and the selfloop a_4 satisfies $\partial a_4 = \partial a_1 + \partial a_2$. If $u \neq w$, then there exists a selfloop arc a_5 of G^{*_1} such that $\partial a_5 = 2\nu u$ since $P \equiv (u, a_1, v, a_3, v, a_1, u)$ is a directed closed path in G^{*_1} with $\partial P = 2\nu u$. Hence, by the definition of the operation (2), there is an arc a_6 such that $\partial a_6 = \frac{1}{2}(\partial a_5 + \partial a_4) = \partial a_1 + \partial a_2$. This completes the proof of the present lemma. □

It should be noted here that the order of the operations (1) and (2) cannot be reversed.

We see from Lemma 3.8 that the transitive closure G^* of a bidirected graph G can be obtained by finding the (1)-transitive closure G^{*_1} of G and then finding the (2)-transitive closure $(G^{*_1})^{*_2}$ of G^{*_1}. Trivially, the (2)-transitive closure of a bidirected graph can be constructed in $\mathrm{O}(|V|^2)$ time. So, let us consider the (1)-transitive closures of bidirected graphs.

Lemma 3.9: *Let G be a bidirected graph. We have*

$$\widetilde{G^{*_1}} = (\widetilde{G})^*, \tag{3.28}$$

where $\cdot^*$ in the right-hand side denotes the transitive closure operator for ordinary directed graphs.

(Proof) We show that there is an arc $\tilde{a}$ in $\widetilde{G^{*_1}}$ such that $\tilde{\partial}\tilde{a} = (u, \sigma(u)) - (w, -\sigma(w))$ if and only if there is an arc $\tilde{b}$ of $(\widetilde{G})^*$ such that $\tilde{\partial}\tilde{b} = (u, \sigma(u)) - (w, -\sigma(w))$.

Suppose that $\tilde{a}$ is an arc of $\widetilde{G^{*_1}}$ such that $\tilde{\partial}\tilde{a} = (u, \sigma(u)) - (w, -\sigma(w))$. Then, there exists an arc a of G^{*_1} such that $\partial a = \sigma(u)u + \sigma(w)w$. By the definition of the (1)-transitive closure, there exists a directed path $P = (v_0(= u), a_1, v_1, \cdots, a_k, v_k(= w))$

of G such that $\partial P = \sigma(u)u + \sigma(w)w$. It follows from Lemma 3.1 that there exists a directed path $\widetilde{P}_1$ in $\widetilde{G}$ such that (3.4). Then, there is an arc $\tilde{b}$ in $(\widetilde{G})^*$ such that $\tilde{\partial}\tilde{b} = (u, \sigma(u)) - (w, -\sigma(w))$.

Conversely, suppose that $\tilde{b}$ is an arc of $(\widetilde{G})^*$ such that $\tilde{\partial}\tilde{b} = (u, \sigma(u)) - (w, -\sigma(w))$. By the definition of the transitive closure $(\widetilde{G})^*$, there exists a directed path

$$\tilde{P}_1 = ((v_0, \sigma_0), a_1^{(\tau_1)}, (v_1, \sigma_1), \cdots, (v_{k-1}, \sigma_{k-1}), a_k^{(\tau_k)}, (v_k, -\sigma_k)), \tag{3.29}$$

in $\widetilde{G}$, where $v_0 = u, \sigma_0 = \sigma(u), v_k = w$ and $\sigma_k = \sigma(w)$. Then, by the definition of $\widetilde{G}$

$$\tilde{P}_2 = ((v_k, \sigma_k), a_k^{(-\tau_k)}, (v_{k-1}, -\sigma_{k-1}), \cdots, (v_1, -\sigma_1), a_1^{(-\tau_1)}, (v_0, -\sigma_0)) \tag{3.30}$$

is also a directed path in $\widetilde{G}$. It follows from Lemma 3.1 that there is a directed path P in G such that $\partial P = \sigma(u)u + \sigma(w)w$. Therefore, there is an arc a in G^{*_1} such that $\partial a = \sigma(u)u + \sigma(w)w$, and hence, there is an arc $\tilde{a}$ in $\widetilde{G^{*_1}}$ such that $\tilde{\partial}\tilde{a} = (u, \sigma(u)) - (w, -\sigma(w))$. □

It follows from Lemma 3.9 that the problem of finding the (1)-transitive closure of a bidirected graph can be reduced to the problem of finding the transitive closure of an ordinary directed graph. It is known that the transitive closure of a directed graph can be found in $O(n^3)$ time, where n in the number of vertices of the directed graph. Hence, the transitive closure of a bidirected $G = (V, A; \partial)$ can also be found in $O(|V|^3)$ time.

For an acyclic bidirected graph $G = (V, A; \partial)$ there exists a unique minimal bidirected subgraph of G whose transitive closure is the same as that of G. This is called *the Hasse diagram* of G and is denoted by $\mathcal{H}(G)$ (see [73], [28, Theorem 2.1]). See Figure 3.2.b.

For a bidirected graph $G = (V, A; \partial)$ an arc $a \in A$ is said to be *redundant* if ∂a can be expressed as a nonnegative linear combination of the other $\partial a'$ ($a' \in A - \{a\}$). Let $B \subseteq A$ be the set of all the redundant arcs of G. The Hasse diagram of G is given by $\mathcal{H}(G) = (V, A - B; \partial)$. If G has a redundant arc, we say G is *redundant*. If G is not redundant, then G is *irredundant*.

Let us consider how we can efficiently compute the Hasse diagram of a bidirected graph. We shall make use of the signed covering graph again.

Lemma 3.10: *Suppose that $\tilde{G} = (\tilde{V}, \tilde{A}; \tilde{\partial})$ is the signed covering graph of a bidirected graph $G = (V, A; \partial)$. Then the Hasse diagram $\mathcal{H}(\tilde{G})$ is self-dual.*

(Proof) Let $\tilde{a}$ be an arc of $\mathcal{H}(\tilde{G})$ such that $\tilde{\partial}\tilde{a} = (v_0, \sigma_0) - (v_k, -\sigma_k)$. Since $\tilde{a}$ is an arc of $\tilde{G}$, there is an arc $\tilde{a}^* \in \tilde{A}$ such that $\tilde{\partial}\tilde{a}^* = (v_k, \sigma_k) - (v_0, -\sigma_0)$. If $\tilde{a}^*$ is not an arc of $\mathcal{H}(\tilde{G})$, then $\tilde{a}^*$ is redundant in $\mathcal{H}(\tilde{G})$. It follows that there is a directed path

$$\tilde{P}_2 = ((v_k, \sigma_k), a_k^{(-\tau_k)}, (v_{k-1}, -\sigma_{k-1}), \cdots, (v_1, -\sigma_1), a_1^{(-\tau_1)}, (v_0, -\sigma_0)) \tag{3.31}$$

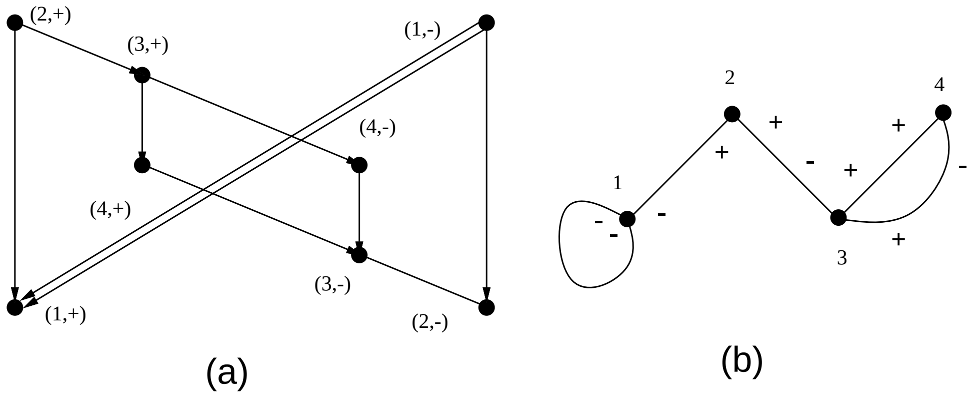

Figure 3.4: $\mathcal{H}(\tilde{G})$ and the corresponding bidirected graph G' of the bidirected graph in Figure 1.2.

in $\mathcal{H}(G)$. By the self-duality of $\tilde{G}$ there exists a directed path

$$\tilde{P}_1 = ((v_0, \sigma_0), a_1^{(\tau_1)}, (v_1, \sigma_1) \cdots, (v_{k-1}, \sigma_{k-1}), a_k^{(\tau_k)}, (v_k, -\sigma_k)) \tag{3.32}$$

in $\tilde{G}$. This means that $\tilde{a}$ is redundant in $\mathcal{H}(G)$ contradicting the assumption. Hence, $\tilde{a}^*$ is an arc of $\mathcal{H}(G)$. □

Let us denote by $G' = (V, A'; \partial)$ the bidirected subgraph of G corresponding to $\mathcal{H}(\tilde{G})$. However, G' may contain an irredundant arc. See Figure 3.4.b. We must remove more arcs from G'. Let $G'' = (V, A''; \partial)$ be the bidirected subgraph of G' obtained by removing all the nonselfloop arcs $a_3 \in A'$ such that for some selfloop arcs a_1 and a_2 of G^{*_1} we have $2\partial a_3 = \partial a_1 + \partial a_2$.

We claim that the removed arcs are all redundant in G. Suppose that $a \in A'$ is removed from G' while constructing G'', i.e., there exist two selfloop arcs a' and a'' in G^{*_1} such that $2\partial a = \partial a' + \partial a''$. It follows from the definition of G^{*_1} that there exist two closed directed paths $C_1 = (v, a_1', \cdots, a_k', v)$ and $C_2 = (w, a_1'', \cdots, a_l'', w)$ in G. such that $\partial C_1 = \partial a'$ and $\partial C_2 = \partial a''$. Hence, we have $\partial a = \frac{1}{2}\partial C_1 + \frac{1}{2}\partial C_2$.

Lemma 3.11: *G'' is the Hasse diagram $\mathcal{H}(G)$ of G.*
(Proof) It suffices to show that G'' is irredundant. Suppose, on the contrary, that there exists a redundant arc a in A''. Then, by the definition, there exist arcs $a_1, \cdots, a_k \in A'' - \{a\}$ such that

$$\partial a = \sum_{i=1}^{k} \lambda_i \partial a_i, \ \lambda_i > 0 \ (i = 1, \cdots, k). \tag{3.33}$$

Define a flow $\varphi\colon A'' \to \mathbf{R}$ in G'' by

$$\varphi(b) = \begin{cases} -1 & (b = a) \\ \lambda_i & (b = a_i) \\ 0 & (\text{otherwise}) \end{cases} \qquad (b \in A''). \tag{3.34}$$

Since φ is a circulation of G'', φ decomposes into the characteristic vectors of circuits each of which conforms to φ (see Lemma 2.6). Hence, there exists a circuit C of G'' such that $\chi_C(a) < 0$, where χ_C is the characteristic vector of circuit C (see Section 2.3). Let us consider the three cases according to the type of circuit C.

Case (i): C is a balanced circle. In this case, a cannot be a selfloop. Therefore, suppose $\partial a = \sigma v + \tau w$ for some $\sigma, \tau \in \{+, -\}$ and $v, w \in V$ with $v \neq w$. Since χ_C conforms to φ, $C - \{a\}$ form a directed path P in G'' such that $\partial P = \sigma v + \tau w$. Since P is also a directed path in G', by Lemma 3.1, there exists a directed path $\tilde{P}$ in $\widetilde{G'} = \mathcal{H}(\tilde{G})$ such that $\tilde{\partial}\tilde{P} = (v, \sigma) - (w, -\tau)$. However, since a is an arc of G', there is an arc $\tilde{a}$ in $\mathcal{H}(\tilde{G})$ such that $\tilde{\partial}\tilde{a} = (v, \sigma) - (w, -\tau)$. This contradicts the irredundancy of $\mathcal{H}(\tilde{G})$.

Case (ii): C is a union of two unbalanced circles and a simple path connecting them. Suppose that a is not a selfloop. Then, $\partial a = \sigma v + \tau w$ for some $\sigma, \tau \in \{+, -\}$ and $v, w \in V$ with $v \neq w$. If a is contained in one of the unbalanced circles, then $C - \{a\}$ forms a directed path P in G'', and leads to a contradiction in the same way as in Case (i). If a is contained in the simple path, then there exist two directed paths P_1 and P_2 in G'', and hence, in G such that $\partial P_1 = 2\sigma v$ and $\partial P_2 = 2\tau w$. Then, there are two selfloops a_1 and a_2 in G^{*1} such that $\partial a_1 = 2\sigma v$ and $\partial a_2 = 2\tau$. By the definition of G'' there exists no arc a' in G'' such that $\partial a' = \sigma v + \tau w$, which contradicts the existence of the arc a. Hence, a must be a selfloop. Then, arc a itself is one of the unbalanced circles. Let $\partial a = 2\sigma v$. It follows that $C - \{a\}$ forms a path P in G'' with $\partial P = 2\sigma v$, which leads to a contradiction as in Case (i).

Case (iii): C is a union of two unbalanced circles. We will have a contradiction by the same argument as in Case (ii). □

We see from Lemma 3.11 together with Lemma 3.9 that the problem of finding the Hasse diagram of an acyclic bidirected graph can be reduced to that of finding the transitive closure and the Hasse diagram of an ordinary directed graph, and hence, can be solved in $\mathrm{O}(|V|^3)$ time.

3.4. The Minimum Cost Circulation Problem

Suppose that $\mathcal{N} = (G = (V, A; \partial), c)$ is a bidirected network, where $c\colon A \to \mathbf{R} \cup \{+\infty\}$ is a capacity function. Let us consider the following linear optimization problem:

$$\begin{array}{lll} (MCC) & \text{Minimize} & \sum_{a\in A} \gamma(a)\varphi(a) \\ & \text{subject to} & \varphi \text{ is a feasible circulation in } \mathcal{N}, \end{array} \tag{3.35}$$

where $\gamma \in \mathbf{R}^A$ is a cost function. For Problem (MCC) we associate the following problem on the signed covering network $\tilde{\mathcal{N}} = (\tilde{G} = (\tilde{V}, \tilde{A}; \tilde{\partial}), \tilde{c})$ of $\mathcal{N}$.

$$\begin{array}{lll} (\widetilde{MCC}) & \text{Minimize} & \sum_{\tilde{a}\in \tilde{A}} \tilde{\gamma}(\tilde{a})\tilde{\varphi}(\tilde{a}) \\ & \text{subject to} & \tilde{\varphi} \text{ is a feasible circulation in } \tilde{\mathcal{N}}, \end{array} \tag{3.36}$$

where $\tilde{\gamma} \in \mathbf{R}^{\tilde{A}}$ is defined as

$$\tilde{\gamma}(a^{(+)}) = \tilde{\gamma}(a^{(-)}) = \gamma(a) \quad (a \in A). \tag{3.37}$$

Given a feasible circulation $\varphi\colon A \to \mathbf{R}$ in a bidirected network $\mathcal{N} = (G = (V, A), c)$. Define the *auxiliary network* $\mathcal{N}_\varphi = (G_\varphi = (V, A_\varphi; \partial_\varphi), c_\varphi, \gamma_\varphi)$ *of* $\mathcal{N}$ with respect to φ as follows.

$$A_\varphi = A_\varphi^+ \cup A_\varphi^-, \tag{3.38}$$

where

$$A_\varphi^+ = \{a \mid a \in A, \varphi(a) < c(a)\},\ \ A_\varphi^- = \{\bar{a} \mid a \in A, 0 < \varphi(a)\}, \tag{3.39}$$

$$\begin{aligned} \partial_\varphi a &= \partial a \quad (a \in A_\varphi^+), && (3.40) \\ \partial_\varphi \bar{a} &= -\partial a \quad (\bar{a} \in A_\varphi^-), && (3.41) \\ c_\varphi(a) &= c(a) - \varphi(a) \quad (a \in A_\varphi^+), && (3.42) \\ c_\varphi(\bar{a}) &= \varphi(a) \quad (\bar{a} \in A_\varphi^-), && (3.43) \\ \gamma_\varphi(a) &= \gamma(a) \quad (a \in A_\varphi^+), && (3.44) \\ \gamma_\varphi(\bar{a}) &= -\gamma(a) \quad (\bar{a} \in A_\varphi^-). && (3.45) \end{aligned}$$

The *cost* $\gamma_\varphi(C)$ of a cycle $C = (v_0, a_1, v_1, \cdots, a_k, v_k)$ in $\mathcal{N}_\varphi$ is defined by $\gamma_\varphi(C) = \sum_{i=1}^k \gamma_\varphi(a_i)$. A *negative cycle* is a cycle with its cost being negative.

An optimality condition for Problem (MCC) is described in terms of the auxiliary graph as follows.

Lemma 3.12: *A feasible circulation φ in $\mathcal{N}$ is an optimal solution of (MCC) if and only if there is no negative cycle in $\mathcal{N}_\varphi$.*

(Proof) If there exists a negative cycle C in $\mathcal{N}_\varphi$, then we have an another feasible circulation φ' with smaller cost than φ by augmenting sufficiently small amount of flow along C. Hence, φ is not an optimal solution of (MCC). Conversely, suppose that there exists no negative cycle in $\mathcal{N}_\varphi$. Let φ' be an optimal solution of (MCC). Since $\varphi' - \varphi$ is a (not necessarily feasible) circulation in G, it follows from Lemma 2.6 that $\varphi' - \varphi$ decomposes as

$$\varphi' - \varphi = \sum_{i=1}^{k} \lambda_i \chi_{C^i}, \quad \lambda_i > 0 \ (i = 1, \cdots, k), \tag{3.46}$$

where χ_{C^i} is the characteristic vector of a circuit C^i in G conforming to $\varphi - \varphi'$ $(i = 1, \cdots, k)$. For each $i = 1, \cdots, k$ define $\chi_{D^i} : A_\varphi \to \{0, 1, 2\}$ by

$$\chi_{D^i}(a_\varphi) = \begin{cases} \chi_{C^i}(a) & (\text{if } a_\varphi = a,\ a \in A,\ \chi_{C^i}(a) > 0) \\ -\chi_{C^i}(a) & (\text{if } a_\varphi = \bar{a},\ a \in A,\ \chi_{C^i}(a) < 0) \\ 0 & (\text{otherwise}). \end{cases} \tag{3.47}$$

We can easily see that for each $i = 1, \cdots, k$ χ_{D^i} is the characteristic vector of a circuit D^i of G_φ and that for each $i = 1, \cdots, k$ D^i naturally induces a cycle E^i with its cost $\gamma_\varphi(E^i)$ given by

$$\begin{aligned} \gamma_\varphi(E^i) &= \sum_{a_\varphi \in D^i} \gamma_\varphi(a_\varphi) \chi_{D^i}(a_\varphi) \\ &= \sum_{a \in C^i} \gamma(a) \chi_{C^i}(a). \end{aligned} \tag{3.48}$$

Since $\mathcal{N}_\varphi$ has no negative cycle, we must have $\sum_{a \in C^i} \gamma(a)\chi_{C^i}(a) \geq 0$ for each $i = 1, \cdots, k$. Then, we have

$$\begin{aligned} \sum_{a \in A} \gamma(a)\varphi'(a) - \sum_{a \in A} \gamma(a)\varphi(a) &= \sum_{a \in A} \gamma(a)(\varphi'(a) - \varphi(a)) \\ &= \sum_{a \in A} \gamma(a) \sum_{C^i \ni a} \lambda_i \chi_{C^i}(a) \\ &= \sum_{i=1}^{k} \lambda_i \sum_{a \in C^i} \gamma(a)\chi_{C^i}(a) \\ &\geq 0, \end{aligned} \tag{3.49}$$

from which follows that φ is also an optimal solution for (MCC). □

Note that the argument in the proof of Lemma 3.12 is almost the same as in the case for the minimum cost circulation in an ordinary directed graph, where Rockafellar's theorem (Theorem 2.6) is fundamental.

Lemma 3.13: *Suppose that $G = (V, A; \partial)$ is a bidirected graph with a cost function $\gamma\colon A \to \mathbf{R}$ and that $\tilde{G} = (\tilde{V}, \tilde{A}; \tilde{\partial})$ is the signed covering graph of G with cost function $\tilde{\gamma}\colon \tilde{A} \to \mathbf{R}$ defined by* (3.37). *Then, G has no negative cycle if and only if there is no negative cycle in the signed covering graph $\tilde{G}$.*
(Proof) Suppose that there is a negative cycle

$$\tilde{C} = ((v_0, \sigma_0), a_1^{(\tau_1)}, (v_1, \sigma_1), \cdots, a_k^{(\tau_k)}, (v_k, \sigma_k)) \tag{3.50}$$

in $\tilde{G}$, where $(v_0, \sigma_0) = (v_k, \sigma_k)$ It follows from Lemma 3.1 that the sequence

$$C = (v_0, a_1, v_1, \cdots, a_k, v_k) \tag{3.51}$$

forms a cycle in G since we have $\partial C = 0$. Moreover, since we have $\gamma(C) = \tilde{\gamma}(\tilde{C})$, the cost of C is negative.

Conversely, if C defined by (3.51) is a negative cycle, then the corresponding sequence $\tilde{C}$ defined by (3.50) is a cycle in $\tilde{G}$ due to Lemma 3.1 and its cost is negative as $\gamma(C) = \tilde{\gamma}(\tilde{C})$. □

Theorem 3.14: *If φ is an optimal solution for Problem (MCC), then $\tilde{\varphi}$ defined by* (3.8) *is an optimal solution for Problem $(\widetilde{MCC})$. Conversely, if $\tilde{\varphi}$ is an optimal solution for Problem $(\widetilde{MCC})$, then φ defined by* (3.6) *is an optimal solution for Problem (MCC). Especially, (MCC) has an optimal solution if and only if $(\widetilde{MCC})$ has.*
(Proof) Suppose that φ is an optimal solution of Problem (MCC). By Lemma 3.12, there is no negative cycle in $\mathcal{N}_\varphi$. It follows from Lemma 3.13 that there is no negative cycle in $\widetilde{\mathcal{N}_\varphi} = \tilde{\mathcal{N}}_{\tilde{\varphi}}$. Hence, by Lemma 3.12, $\tilde{\varphi}$ is an optimal solution for Problem $(\widetilde{MCC})$.

Conversely, suppose that $\tilde{\varphi}$ is an optimal solution for Problem $(\widetilde{MCC})$. Define a feasible circulation $\tilde{\varphi}'$ by

$$\tilde{\varphi}'(a^{(\epsilon)}) = \tilde{\varphi}(a^{(-\epsilon)}) \quad (a^{(\epsilon)} \in \tilde{A}). \tag{3.52}$$

One can easily see that $\tilde{\varphi}'$ is a feasible circulation, and indeed, is optimal. Now, $\tilde{\varphi}'' = \frac{1}{2}(\tilde{\varphi} + \tilde{\varphi}')$ is also an optimal circulation. Hence, there is no negative cycle in $\tilde{\mathcal{N}}_{\tilde{\varphi}''} = \widetilde{\mathcal{N}_{\varphi''}}$, where φ'' is defined by (3.6) with φ being replaced by φ''. It follows from Lemma 3.13 that there is no negative cycle in $\mathcal{N}_{\varphi''}$. Therefore, $\varphi = \varphi''$ is an optimal solution for Problem (MCC). □

Hence, Problem (MCC) can be reduced to an ordinary minimum cost circulation problem $(\widetilde{MCC})$ and can be solved in polynomial time. However, even if the capacity function is integral, there may not exist an integral optimal solution for (MCC) as is shown in the following example: $G = (\{v_1, v_2\}, \{a_1, a_2, a_3\}; \partial)$, where $\partial a_1 = 2v_1$, $\partial a_2 = v_2 - v_1$, $\partial a_3 = -2v_2$, $c(a_1) = c(a_2) = c(a_3) = 1$ and $\gamma(a_2) = 1$, $\gamma(a_1) = \gamma(a_3) = 0$. Still, we have the following.

Corollary 3.15: *If Problem* (MCC) *has an optimal solution and* c *is integral (respectively, even), then it has a half-integral (respectively, integral) optimal solution.*
(Proof) If (MCC) has an optimal solution, then by Theorem 3.14 $(\widetilde{MCC})$ also has an optimal solution. Furthermore, if c is integral (even), then $(\widetilde{MCC})$ has an integral (even) optimal solution. Again, by Theorem 3.14, there exist a half-integral (integral) optimal solution for (MCC). □

3.5. The Minimum Cut Problem and the Maximum Flow Problem

In this section we consider two optimization problems on bidirected graphs, namely, the minimum cut problem and the maximum flow problem. It will be shown that the problems can be reduced to those on signed covering graphs in linear time and that the well-known maxflow-mincut theorem generalizes to bidirected graphs.

Suppose that $\mathcal{N} = (G = (V, A; \partial), c)$ is a bidirected network and s, t are not necessarily distinct two vertices of G. We have three types of minimum cut problem. A *minimum cut problem* is any one of $(MC1) \sim (MC3)$ defined as follows:

$$(MC1) \quad \text{Minimize} \quad \kappa_c(X, Y) \qquad \text{subject to} \quad (X, Y) \in 3^V,\ s, t \in X. \tag{3.53}$$

$$(MC2) \quad \text{Minimize} \quad \kappa_c(X, Y) \qquad \text{subject to} \quad (X, Y) \in 3^V,\ s, t \in Y. \tag{3.54}$$

$$(MC3) \quad \text{Minimize} \quad \kappa_c(X, Y) \qquad \text{subject to} \quad (X, Y) \in 3^V,\ s \in X,\ t \in Y. \tag{3.55}$$

This variety of problems corresponds to the three types of arc in a bidirected graph. If we have $s = t$ in $(MC3)$, the problem is obviously infeasible. Hence, we assume $s \neq t$ when Problem $(MC3)$ is considered.

Associated with Problems $(MC1) \sim (MC3)$, let us consider the following problems $(\widetilde{MC}1) \sim (\widetilde{MC}3)$ on the signed covering network $\widetilde{\mathcal{N}}$ of $\mathcal{N}$.

$$(\widetilde{MC}1) \quad \text{Minimize} \quad \tilde{\kappa}_{\tilde{c}}(\tilde{X}) \qquad \text{subject to} \quad \tilde{X} \in 2^{\tilde{V}}, (s, +), (t, +) \in \tilde{X},\ (s, -), (t, -) \notin \tilde{X}. \tag{3.56}$$

$$(\widetilde{MC}2) \quad \text{Minimize} \quad \tilde{\kappa}_{\tilde{c}}(\tilde{X}) \qquad \text{subject to} \quad \tilde{X} \in 2^{\tilde{V}}, (s, -), (t, -) \in \tilde{X},\ (s, +), (t, +) \notin \tilde{X}. \tag{3.57}$$

$$
\begin{array}{lll}
(\widetilde{MC}3) & \text{Minimize} & \tilde{\kappa}_{\tilde{c}}(\tilde{X}) \\
 & \text{subject to} & \tilde{X} \in 2^{\tilde{V}}, (s,+), (t,-) \in \tilde{X},\ (s,-), (t,+) \notin \tilde{X}.
\end{array}
\tag{3.58}
$$

We call a subset $\tilde{X}$ of $\tilde{V}$ *isotropic* if $\tilde{X} \cap \tilde{X}^* = \emptyset$.

Theorem 3.16: *For each $i = 1, 2, 3$ there exists a one-to-one correspondence between the set of the optimal solutions for (MCi) and the set of the isotropic optimal solutions for $(\widetilde{MC}i)$.*

(Proof) It suffices to show only the case when $i = 3$ since the other cases can be treated similarly.

If $(X, Y) \in 3^V$ is an optimal solution of Problem $(MC3)$, then we have that

$$
\tilde{X} = (X, +) \cup (Y, -) \tag{3.59}
$$

is a feasible solution of Problem $(\widetilde{MC}3)$ and from Lemma 3.3 that the optimal value of $(\widetilde{MC}3)$ is not greater than that of $(MC3)$. Conversely, for any optimal solution $\tilde{X}$ of $(\widetilde{MC}3)$ $\tilde{X}' = \tilde{X} - (\tilde{X} \cap \tilde{X}^*)$ is also an optimal solution of $(\widetilde{MC}3)$ by Lemma 3.4. The signed subset (X, Y) defined by

$$
(X, Y) = (\{v \mid (v, +) \in \tilde{X}'\}, \{v \mid (v, -) \in \tilde{X}'\}) \tag{3.60}
$$

gives a feasible solution of $(MC3)$. Hence, by Lemma 3.3, the optimal value of Problem $(\widetilde{MC}3)$ is greater than or equal to that of $(MC3)$. Therefore, the two optimal values are equal. Now, it is easy to see the mapping defined by (3.59) (or (3.60)) gives a desired one-to-one correspondence. □

We see from Theorem 3.16 that a minimum cut problem for a bidirected graph can be reduced to a minimum cut problem for an ordinary directed graph after contractions, if necessary, of two pairs of vertices.

Given a bidirected network $\mathcal{N} = (G = (V, A; \partial), c)$ and not necessarily distinct two vertices s and t, let us consider the *maximum flow problem* defined as any of the following $(MF1) \sim (MF3)$:

$$
\begin{array}{lll}
(MF1) & \text{Maximize} & \partial\varphi(\{s, t\}) \\
 & \text{subject to} & \partial\varphi(v) = 0 \quad (v \in V - \{s, t\}), \\
 & & 0 \le \varphi(a) \le c(a) \quad (a \in A).
\end{array}
\tag{3.61}
$$

$$
\begin{array}{lll}
(MF2) & \text{Maximize} & -\partial\varphi(\{s, t\}) \\
 & \text{subject to} & \partial\varphi(v) = 0 \quad (v \in V - \{s, t\}), \\
 & & 0 \le \varphi(a) \le c(a) \quad (a \in A).
\end{array}
\tag{3.62}
$$

$$\begin{array}{llll} (MF3) & \text{Maximize} & \partial\varphi(s) - \partial\varphi(t) & \\ & \text{subject to} & \partial\varphi(v) = 0 \quad (v \in V - \{s,t\}), & (3.63) \\ & & 0 \le \varphi(a) \le c(a) \quad (a \in A). & \end{array}$$

Note that the maximum flow problem for ordinary directed networks is of $(MF3)$ type. As in the case for directed networks, the maximum flow problem for bidirected graphs can be considered as a minimum cost circulation problem: for Problem $(MF3)$ if we add to $\mathcal{N}$ a new arc $a_0 \notin A$ with its boundary $\partial a_0 = -s+t$, then $(MF3)$ is just a minimum cost circulation problem with the cost function γ, say, $\gamma(a_0) = -2$ and $\gamma(a) = 0$ $(a \in A)$. Similarly, $(MF1)$ and $(MF2)$ reduce to minimum cost circulation problems if we add a new arc a_0 with $\partial a_0 = -s-t$ and $\partial a_0 = s+t$, respectively. Therefore, the maximum flow problem on a bidirected graph can be reduced to a maximum flow problem on its signed covering graph as was shown in Section 3.4.

The duality between the maximum flow problems $(MF1) \sim (MF3)$ and the minimum cut problems $(MC1) \sim (MC3)$ can be seen as follows. It is easy to see that for any feasible solution φ of $(MF3)$ and any feasible solution (X,Y) of $(MC3)$ we have $\partial\varphi(s) - \partial\varphi(t) \le \kappa_c(X,Y)$. Also, the similar (weak) dualities hold between $(MF1)$ and $(MC1)$, and between $(MF2)$ and $(MC2)$.

Theorem 3.17: *For each $i = 1,2,3$ the maximum value of Problem (MFi) is equal to the minimum value of Problem (MCi).*

(Proof) It suffices to consider Problem $(MC3)$ only since the other problems can be treated similarly. Let (X,Y) be an optimal solution for Problem $(MC3)$. It follows from Theorem 3.16 that $\tilde{X}$ defined by (3.59) is an optimal solution for problem $(\widetilde{MC}3)$ and that $\kappa_c(X,Y) = \tilde{\kappa}_c(\tilde{X})$. Then, by the maxflow-mincut theorem for ordinary directed graphs, there exists a feasible flow $\tilde{\varphi}$ in the signed covering graph $\tilde{\mathcal{N}}$ such that

$$\tilde{\partial}\tilde{\varphi}(v,\pm) = 0 \quad (v \in V - \{s,t\}) \tag{3.64}$$

and

$$\begin{aligned} \tilde{\kappa}_c(\tilde{X}) &= \tilde{\partial}\tilde{\varphi}(s,+) + \tilde{\partial}\tilde{\varphi}(t,-) \\ &= -\tilde{\partial}\tilde{\varphi}(s,-) - \tilde{\partial}\tilde{\varphi}(t,+). \end{aligned} \tag{3.65}$$

It follows from Lemma 3.2 that φ defined by (3.6) is a feasible flow in $\mathcal{N}$ such that

$$\partial\varphi(s) - \partial\varphi(t) = \tilde{\kappa}_c(\tilde{X}) \tag{3.66}$$

and $\partial\varphi(v) = 0$ for $v \in V - \{s,t\}$. □

Again, the integrality for a maximum flow in ordinary directed networks is no longer holds for bidirected networks.

3.6. The Minimum-Weight Ideal Problem

In this section we consider the minimum-weight ideal problem for bidirected graphs and show that the same reduction technique is valid as well as the problems considered in the previous sections.

Given a bidirected graph $G = (V, A; \partial)$ and a weight function $w : V \to \mathbf{R}$, the *minimum-weight ideal problem* is defined as follows:

$$\begin{array}{lll} (MWI) & \text{Minimize} & w(X,Y) = \sum_{v \in X} w(v) - \sum_{v \in Y} w(v) \\ & \text{subject to} & (X,Y) \in \mathcal{I}(G), \end{array} \tag{3.67}$$

where $\mathcal{I}(G)$ is the set of all the ideals of the bidirected graph $G = (V, A; \partial)$.

Let us consider the bidirected network $\mathcal{N} = (G, c)$, where the capacity function c is defined as

$$c(a) = +\infty \qquad (a \in A). \tag{3.68}$$

Then, it follows from the definition of ideal and (3.68) that

$$\mathcal{I}(G) = \{(X,Y) \mid (X,Y) \in 3^V, \kappa_c(X,Y) = 0\}, \tag{3.69}$$

where κ_c: $3^V \to \mathbf{R} \cup \{+\infty\}$ is the cut function associated with $\mathcal{N}$.

Since $\kappa_c(X,Y) \geq 0$ for any $(X,Y) \in 3^V$, it follows from Lemma 2.2 that

Lemma 3.18 ([10]): *For any bidirected graph $G = (V, A; \partial)$ $\mathcal{I}(G)$ is a $\{\sqcup, \sqcap\}$-closed family on V with $(\emptyset, \emptyset) \in \mathcal{I}(G)$.* □

Let us consider the signed covering graph $\tilde{G} = (\tilde{V}, \tilde{A}; \tilde{\partial})$ of a bidirected graph $G = (V, A; \partial)$. Recall that a subset $\tilde{J}$ of $\tilde{V}$ is an order ideal of $\tilde{G}(\mathcal{P})$ if for each arc $\tilde{a} \in \tilde{A}$ with $\tilde{\partial}\tilde{a} = (w, \tau) - (v, \sigma)$ and $(w, \tau) \in \tilde{J}$ we have $(v, \sigma) \in \tilde{J}$.

Let us consider a capacity function $\tilde{c}$ defined by $\tilde{c}(\tilde{a}) = +\infty$ $(\tilde{a} \in \tilde{A})$. Then, the set $\mathcal{I}_+(\tilde{G})$ of all the order ideals of $\tilde{G}$ is given as

$$\mathcal{I}_+(\tilde{G}) = \{\tilde{J} \mid \tilde{J} \subseteq \tilde{V}, \tilde{\kappa}_{\tilde{c}}(\tilde{J}) = 0\}, \tag{3.70}$$

where $\tilde{\kappa}_{\tilde{c}}$ is the cut function associated with $\widetilde{\mathcal{N}} = (\tilde{G}, \tilde{c})$. We see from Lemma 3.3, (3.69) and (3.70) the following lemma.

Lemma 3.19 (Fischer [31], Ando, Fujishige and Nemoto [11]): *The mapping*

$$(X,Y) \mapsto (X,+) \cup (Y,-) \tag{3.71}$$

gives a one-to-one correspondence between the set of the ideals of G and the set of the isotropic order ideals of $\tilde{G}$. □

Now, define a weight function $\tilde{w}\colon \tilde{V} \to \mathbf{R}$ by

$$\tilde{w}(v,+) = w(v) \quad ((v,+) \in \tilde{V}), \tag{3.72}$$

$$\tilde{w}(v,-) = -w(v) \quad ((v,-) \in \tilde{V}). \tag{3.73}$$

It follows from Lemmas 3.19 and 3.4 that Problem (MWI) is reduced to the problem of finding a minimum-weight order ideal of the directed graph $\tilde{G}$ with respect to the weight $\tilde{w}$ ([11]). That is, to find a minimum-weight ideal of the bidirected graph G we first compute a minimum-weight order ideal $\tilde{J}$ of the directed graph $\tilde{G}$ by any minimum-cut algorithm (see [65], [66] and [1]), then obtain the isotropic order ideal $\tilde{J}' \leftarrow \tilde{J} - \tilde{J}^*$ by (3.16), and find the ideal (X, Y) of G corresponding to $\tilde{J}'$ by (3.71). Note that the weight of an order ideal $\tilde{J}$ of $\tilde{G}$ is equal to that of its corresponding isotropic order ideal $\tilde{J}' = \tilde{J} - \tilde{J}^*$.

Chapter 4

$\{\sqcup, \sqcap\}$-closed Families and Signed Posets

A bidirected graph $G = (V, A; \partial)$ is called a *signed poset* on V if it is acyclic and transitively closed, i.e., it satisfies the following:

(i) There are no two arcs $a_1, a_2 \in A$ such that $\partial a_1 = -\partial a_2$.

(ii) For any two arcs a_1, a_2 in G that are oppositely incident to a vertex there exists an arc $a_3 \in A$ such that $\partial a_3 = \partial a_1 + \partial a_2$.

(iii) For any two selfloops a_1, a_2 in G incident to distinct vertices there exists an arc $a_3 \in A$ such that $2\partial a_3 = \partial a_1 + \partial a_2$.

See Figure 3.2.a for an example of a signed poset. V. Reiner [73] has introduced the concept of signed poset and showed the so-called signed Birkhoff theorem that is a signed analogue of the well-known Birkhoff theorem on the relationship between the set of ideals of a poset and a distributive lattice. (Our definition of signed poset is equivalent but slightly different to the one given by Reiner [73].) The signed Birkhoff theorem [73, Theorem 4.8] asserts that for a finite lattice $\mathcal{L}$ with the maximum element $\hat{1}$, $\mathcal{L} - \{\hat{1}\}$ is isomorphic to the set of ideals of some signed poset $\mathcal{P}$ if and only if $\mathcal{L}$ is B_n-distributive, where $\mathcal{P}$ is determined by $\mathcal{L}$ up to isomorphism as a signed poset (see [73] for the terminology).

We shall show a theorem that there exists a one-to-one correspondence between the set of all the simple and spanning $\{\sqcup, \sqcap\}$-closed families $\mathcal{F} \subseteq 3^V$ on V and the set of all the signed posets $\mathcal{P}$ on V such that each such $\mathcal{F}$ is the set of all the ideals of the corresponding signed poset $\mathcal{P}$. The theorem is a set-theoretical version of the signed Birkhoff theorem of V. Reiner, which sheds a new light on the signed Birkhoff theorem. We also discuss the representations of general non-simple or non-spanning $\{\sqcup, \sqcap\}$-closed families and make clear the relationship between balanced bidirected graphs and proper bisubmodular systems.

The result obtained in this chapter gives an important basis for developing a theory of bisubmodular functions and associated polyhedra since for any point x in the bisubmodular polyhedron defined by (2.15) the collection $\mathcal{F}(x)$ of *tight* signed sets given by

$$\mathcal{F}(x) = \{(X, Y) \mid (X, Y) \in \mathcal{F}, x(X, Y) = f(X, Y)\}$$

forms a $\{\sqcup, \sqcap\}$-closed subfamily of $\mathcal{F}$. We can see that a point $x \in \mathrm{P}_*(f)$ is an extreme point of $\mathrm{P}_*(f)$ if and only if $\mathcal{F}(x)$ is simple and spanning (see Chapter 5). Therefore, for each extreme point x of the bisubmodular polyhedron $\mathrm{P}_*(f)$ we have a signed poset associated with $\mathcal{F}(x)$. We can characterize the adjacency of extreme points of the bisubmodular polyhedron in terms of the Hasse diagrams of the associated signed posets. By means of the signed poset representation we can further examine facets, faces, dimensions, connected components, the membership problem etc. for bisubmodular polyhedra (see Chapter 5).

This chapter is based on the paper [4] and partially on [8].

4.1. $\{\sqcup, \sqcap\}$-closed Families and their Representations

For any bidirected graph $G = (V, A; \partial)$ let us denote the set $\{\partial a \mid a \in A\}$ by ∂A.

Now, let us consider a simple and spanning $\{\sqcup, \sqcap\}$-closed family $\mathcal{F} \subseteq 3^V$. (Recall the definition of a simple and spanning $\{\sqcup, \sqcap\}$-closed family given in Section 2.1).

For each $v \in V$ define

$$\mathrm{F}(+v) = \sqcap\{(X, Y) \mid v \in X, (X, Y) \in \mathcal{F}\} \tag{4.1}$$

if there exists some $(X, Y) \in \mathcal{F}$ such that $v \in X$, and define

$$\mathrm{F}(-v) = \sqcap\{(X, Y) \mid v \in Y, (X, Y) \in \mathcal{F}\} \tag{4.2}$$

if there exists some $(X, Y) \in \mathcal{F}$ such that $v \in Y$. If there is no $(X, Y) \in \mathcal{F}$ such that $v \in X$ (or $v \in Y$), then we define $\mathrm{F}(+v) = (\emptyset, \emptyset)$ (or $\mathrm{F}(-v) = (\emptyset, \emptyset)$). Note that since $\mathcal{F}$ is a spanning family on V, $\mathrm{F}(+v)$ or $\mathrm{F}(-v)$ is nonnull for any $v \in V$.

For any $Z = (X, Y) \in 3^V$ we define

$$Z^+ = X, \quad Z^- = Y. \tag{4.3}$$

Given a simple and spanning $\{\sqcup, \sqcap\}$-closed family $\mathcal{F} \subseteq 3^V$ on V, we construct a bidirected graph $G(\mathcal{F})$ as follows. $G(\mathcal{F})$ has the vertex set V. The arc set A is constructed by the following procedures (1)$\sim$(3):

(1) For each $v \in V$

- (1a) if $\mathrm{F}(-v) = (\emptyset, \emptyset)$, add a selfloop a at v such that $\partial a = -2v$,
- (1b) if $\mathrm{F}(+v) = (\emptyset, \emptyset)$, add a selfloop a at v such that $\partial a = 2v$.

(2) For each distinct $v, w \in V$

- (2a) if $w \in \mathrm{F}(+v)^+$, add an arc a such that $\partial a = v - w$,

(2b) if $w \in \mathrm{F}(+v)^-$, add an arc a such that $\partial a = v + w$,
(2c) if $w \in \mathrm{F}(-v)^+$, add an arc a such that $\partial a = -v - w$,
(2d) if $w \in \mathrm{F}(-v)^-$, add an arc a such that $\partial a = -v + w$.

(3) For any two selfloops a_1 and a_2 that are incident to distinct vertices, add an arc a_3 such that $2\partial a_3 = \partial a_1 + \partial a_2$.

During the construction of the arc set A, if an arc to be added has already been constructed, then we skip the operation.

To show that the bidirected graph $G(\mathcal{F})$ constructed above is a signed poset, we need some lemmas.

Lemma 4.1: *For any distinct* $v, w \in V$
(a) *if* $w \in \mathrm{F}(+v)^+$, *then* $v \notin \mathrm{F}(+w)^+$,
(b) *if* $w \in \mathrm{F}(+v)^-$, *then* $v \notin \mathrm{F}(-w)^+$,
(c) *if* $w \in \mathrm{F}(-v)^+$, *then* $v \notin \mathrm{F}(+w)^-$,
(d) *if* $w \in \mathrm{F}(-v)^-$, *then* $v \notin \mathrm{F}(-w)^-$.
(Proof) We show (a) (the proofs of the other cases are similar).

Suppose, on the contrary, that $w \in \mathrm{F}(+v)^+$ and $v \in \mathrm{F}(+w)^+$. Since $\mathcal{F}$ is simple, there is some $(X, Y) \in \mathcal{F}$ such that (1) $v \in X \cup Y$ and $w \notin X \cup Y$, or (2) $v \notin X \cup Y$ and $w \in X \cup Y$. In Case (1), if $v \in X$, then $(\mathrm{F}(+v) \sqcap (X, Y))^+$ contains v but not w, which contradicts the minimality of $\mathrm{F}(+v)$; and if $v \in Y$, then $(\mathrm{F}(+w) \sqcup (X, Y))^+$ contains w but not v, which contradicts the minimality of $\mathrm{F}(+w)$. Case (2) can be treated similarly as Case (1). □

Lemma 4.2: *For any distinct* $v, w \in V$
(a) *if* $w \in \mathrm{F}(+v)^+$, *then* $\mathrm{F}(+w) \sqsubset \mathrm{F}(+v)$,
(b) *if* $w \in \mathrm{F}(+v)^-$, *then* $\mathrm{F}(-w) \sqsubset \mathrm{F}(+v)$,
(c) *if* $w \in \mathrm{F}(-v)^+$, *then* $\mathrm{F}(+w) \sqsubset \mathrm{F}(-v)$,
(d) *if* $w \in \mathrm{F}(-v)^-$, *then* $\mathrm{F}(-w) \sqsubset \mathrm{F}(-v)$.
(Proof) If the inclusion $\sqsubset$ is replaced by the inclusion $\sqsubseteq$ with equality, then each assertion easily follows from the definition (the minimality) of $\mathrm{F}(\cdot)$. The strict inclusion $\sqsubset$ is due to Lemma 4.1. □

Lemma 4.3: *For any distinct* $v, w \in V$
(a) *if* $w \in \mathrm{F}(+v)^+$ *and* $\mathrm{F}(-w) \neq (\emptyset, \emptyset)$, *then* $v \in \mathrm{F}(-w)^-$,
(b) *if* $w \in \mathrm{F}(+v)^-$ *and* $\mathrm{F}(+w) \neq (\emptyset, \emptyset)$, *then* $v \in \mathrm{F}(+w)^-$,
(c) *if* $w \in \mathrm{F}(-v)^+$ *and* $\mathrm{F}(-w) \neq (\emptyset, \emptyset)$, *then* $v \in \mathrm{F}(-w)^+$,

(d) *if* $w \in \mathrm{F}(-v)^-$ *and* $\mathrm{F}(+w) \neq (\emptyset, \emptyset)$*, then* $v \in \mathrm{F}(+w)^+$.

(Proof) We show (a) (the proofs of the other cases are similar).

Suppose, on the contrary, that $w \in \mathrm{F}(+v)^+$, $\mathrm{F}(-w) \neq (\emptyset, \emptyset)$ and $v \notin \mathrm{F}(-w)^-$. Then we have $v \in (\mathrm{F}(+v) \sqcup \mathrm{F}(-w))^+$ and $w \notin (\mathrm{F}(+v) \sqcup \mathrm{F}(-w))^+$, which contradicts the minimality of $\mathrm{F}(+v)$. □

Lemma 4.3 partly corresponds to Proposition 4.6 in [73].

Lemma 4.4: *For any distinct* $v, w \in V$

(a) *if* $w \in \mathrm{F}(+v)^+$ *and* $\mathrm{F}(-v) = (\emptyset, \emptyset)$*, then* $\mathrm{F}(-w) = (\emptyset, \emptyset)$,

(b) *if* $w \in \mathrm{F}(+v)^-$ *and* $\mathrm{F}(-v) = (\emptyset, \emptyset)$*, then* $\mathrm{F}(+w) = (\emptyset, \emptyset)$,

(c) *if* $w \in \mathrm{F}(-v)^+$ *and* $\mathrm{F}(+v) = (\emptyset, \emptyset)$*, then* $\mathrm{F}(-w) = (\emptyset, \emptyset)$,

(d) *if* $w \in \mathrm{F}(-v)^-$ *and* $\mathrm{F}(+v) = (\emptyset, \emptyset)$*, then* $\mathrm{F}(+w) = (\emptyset, \emptyset)$.

(Proof) We show (a) (the proofs of the other cases are similar).

Suppose, on the contrary, that $w \in \mathrm{F}(+v)^+$, $\mathrm{F}(-v) = (\emptyset, \emptyset)$ and $\mathrm{F}(-w) \neq (\emptyset, \emptyset)$. Then from Lemma 4.3 we have $v \in \mathrm{F}(-w)^-$, which contradicts the assumption that $\mathrm{F}(-v) = (\emptyset, \emptyset)$. □

From Lemmas 4.1~4.4 we have the following.

Theorem 4.5: *The bidirected graph* $G(\mathcal{F}) = (V, A; \partial)$ *defined above is a signed poset.*

(Proof) Let us check conditions (i)~(iii) in the definition of a signed poset.

(i) Suppose, on the contrary, that there exist two arcs $a_1, a_2 \in A$ such that $\partial a_1 + \partial a_2 = 0$. These a_1 and a_2 are both nonselfloop arcs or both selfloop arcs. Since selfloop arcs are added only by Procedure (1), the case of two selfloops is excluded due to the remark given after the definition of $\mathrm{F}(\cdot)$. Hence, suppose a_1 and a_2 are nonselfloop arcs. By the same reason as above, they are not both added by Procedure (3). Also, they cannot be both added by Procedure (2) due to Lemma 4.1. The only possibility is now the case when one is added by Procedure (2) and the other by Procedure (3). However, this case is also excluded by Lemma 4.4.

(ii) Suppose $\partial a_1 + \partial a_2 = \pm v \pm w$ for some $v, w \in V$. We treat only the case when $\partial a_1 + \partial a_2 = v + w$ since the other cases are treated similarly. Then we have $v - u \in \partial A$ and $u + w \in \partial A$.

When $u = w$, we have $\mathrm{F}(+w) = (\emptyset, \emptyset)$ and hence $v \in \mathrm{F}(-w)^-$. Then, by Lemma 4.4, we have $\mathrm{F}(+v) = (\emptyset, \emptyset)$, which implies $2v \in \partial A$. Hence, Procedure (3) constructs an arc $a_3 \in A$ with $\partial a_3 = v + w$.

When $u \neq w$ and $v \neq w$, this means that we have one of the following (a)~(f): (a) $v \in \mathrm{F}(-u)^-, u \in \mathrm{F}(+w)^-$; (b) $u \in \mathrm{F}(+v)^+, w \in \mathrm{F}(+u)^-$; (c) $v \in \mathrm{F}(-u)^-, w \in \mathrm{F}(+u)^-$;

(d) $u \in \mathrm{F}(+v)^+, u \in \mathrm{F}(+w)^-$; (e) $\mathrm{F}(+v) = (\emptyset,\emptyset), \mathrm{F}(-u) = (\emptyset,\emptyset), w \in \mathrm{F}(+u)^-$; (f) $\mathrm{F}(+w) = (\emptyset,\emptyset), \mathrm{F}(+u) = (\emptyset,\emptyset), v \in \mathrm{F}(-u)^-$. Both (a) and (b) imply $v + w \in \partial A$ from Lemma 4.2. In Case (c), if both $\mathrm{F}(+v) = (\emptyset,\emptyset)$ and $\mathrm{F}(+w) = (\emptyset,\emptyset)$, then we have $v+w \in \partial A$ due to Procedure (3). Moreover, if $\mathrm{F}(+w) \neq (\emptyset,\emptyset)$ (or $\mathrm{F}(+v) \neq (\emptyset,\emptyset)$), then Case (c) is reduced to Case (a) (or Case (b)) from Lemma 4.3. Case (d) is reduced to Case (a) (and Case (b)) from Lemma 4.3 since $\mathrm{F}(-u) \neq (\emptyset,\emptyset)$ (and $\mathrm{F}(+u) \neq (\emptyset,\emptyset)$). In Case (e) (or Case (f)) we have $\mathrm{F}(+w) = (\emptyset,\emptyset)$ (or $\mathrm{F}(+v) = (\emptyset,\emptyset)$) due to Lemma 4.4, so that we have $v + w \in \partial A$ by Procedure (3).

When $v = w$, if there is a selfloop a with $\partial a = 2v$, then we are finished. So, suppose that arcs a_1, a_2 with $\partial a_1 = v - w$ and $\partial a_2 = v + w$ are not constructed by Procedure (3). Then we have (I) $v \in \mathrm{F}(+w)^-$ or $w \in \mathrm{F}(+v)^-$ and (II) $v \in \mathrm{F}(-w)^-$ or $w \in \mathrm{F}(+v)^+$. If we have $\mathrm{F}(+v) \neq (\emptyset,\emptyset)$, from Lemma 4.3 (I) implies $w \in \mathrm{F}(+v)^-$ and (II) implies $w \in \mathrm{F}(+v)^+$, a contradiction. Hence, $\mathrm{F}(+v) = (\emptyset,\emptyset)$.

(iii) This is due to the definition of $G(\mathcal{F})$. □

We now denote the signed poset $G(\mathcal{F})$ by $\mathcal{P}(\mathcal{F}) = (V, A; \partial)$.

Lemma 4.6: *Let $(X,Y) \in 3^V$ be an ideal of the signed poset $\mathcal{P}(\mathcal{F})$. Then we have*

$$(\emptyset,\emptyset) \neq \mathrm{F}(+v) \sqsubseteq (X,Y) \qquad (v \in X) \tag{4.4}$$

$$(\emptyset,\emptyset) \neq \mathrm{F}(-v) \sqsubseteq (X,Y) \qquad (v \in Y) \tag{4.5}$$

(Proof) Since (X,Y) is an ideal of $\mathcal{P}(\mathcal{F})$, relations (4.4) and (4.5) follow from the definition of $\mathcal{P}(\mathcal{F})$. □

Now, we show the following theorem.

Theorem 4.7: *The set of all the ideals of the signed poset $\mathcal{P}(\mathcal{F})$ coincides with the given $\mathcal{F}$.*

(Proof) Suppose that $(X,Y) \in 3^V$ is an ideal of $\mathcal{P}(\mathcal{F})$. From Lemma 4.6 we have

$$(X,Y) = (\sqcup_{v\in X}\mathrm{F}(+v)) \sqcup (\sqcup_{v\in Y}\mathrm{F}(-v)). \tag{4.6}$$

Hence, $(X,Y) \in \mathcal{F}$.

Conversely, suppose $(X,Y) \in \mathcal{F}$. Then we have

$$(\emptyset,\emptyset) \neq \mathrm{F}(+v) \sqsubseteq (X,Y) \qquad (v \in X), \tag{4.7}$$

$$(\emptyset,\emptyset) \neq \mathrm{F}(-v) \sqsubseteq (X,Y) \qquad (v \in Y). \tag{4.8}$$

If (X,Y) is not an ideal of $\mathcal{P}(\mathcal{F})$, then we have the following (I) or (II):

(I) For some $v \in X$ we have one of the following three:

(a) There is a selfloop a such that $\partial a = 2v$.

(b) There are a nonselfloop arc a and a vertex $w \notin Y$ such that $\partial a = v + w$.

(c) There are a nonselfloop arc a and a vertex $w \notin X$ such that $\partial a = v - w$.

(II) For some $v \in Y$ we have one of the following three:

(a) There is a selfloop a such that $\partial a = -2v$.

(b) There are a nonselfloop arc a and a vertex $w \notin X$ such that $\partial a = -v - w$.

(c) There are a nonselfloop arc a and a vertex $w \notin Y$ such that $\partial a = -v + w$.

Case (I-a) is impossible since $\mathrm{F}(+v) \neq (\emptyset, \emptyset)$ for $v \in X$. In Case (I-b), we have $w \in \mathrm{F}(+v)^-$ or $v \in \mathrm{F}(+w)^-$. But $w \in \mathrm{F}(+v)^-$ is impossible from (4.7) since $w \notin Y$. So, we must have $v \in \mathrm{F}(+w)^-$, which implies $w \in \mathrm{F}(+v)^- \subseteq Y$ (due to Lemma 4.3 and (4.7)), a contradiction. Similarly as Case (I-b), Case (I-c) also leads us to a contradiction. Case (II) can be treated similarly as Case (I).

Consequently, (X, Y) must be an ideal of $\mathcal{P}(\mathcal{F})$. □

Theorem 4.7 asserts that $\mathcal{P}(\cdot)$ defines a one-to-one mapping from the set of simple spanning $\{\sqcup,\sqcap\}$-closed families on V to the set of signed posets on V. We show that the mapping is also onto.

For a signed poset $\mathcal{P} = (V, A; \partial)$ and a vertex $v \in V$, when $2v \notin \partial A$, define

$$\mathrm{I}(+v) = (\{w \mid v - w \in \partial A\} \cup \{v\}, \{w \mid v + w \in \partial A\}) \tag{4.9}$$

and when $-2v \notin \partial A$, define

$$\mathrm{I}(-v) = (\{w \mid -v - w \in \partial A\}, \{w \mid -v + w \in \partial A\} \cup \{v\}). \tag{4.10}$$

Also, if $2v \in \partial A$ (or $-2v \in \partial A$), we define $\mathrm{I}(+v) = (\emptyset, \emptyset)$ (or $\mathrm{I}(-v) = (\emptyset, \emptyset)$). We can easily see that $\mathrm{I}(+v)$ and $\mathrm{I}(-v)$ are ideals of $\mathcal{P}$. We call $\mathrm{I}(+v)$ the *positive principal ideal* at v and $\mathrm{I}(-v)$ the *negative principal ideal* at v of $\mathcal{P}$. (In fact, for a simple and spanning $\{\sqcup,\sqcap\}$-closed family $\mathcal{F}$ on V and its corresponding signed poset $\mathcal{P}$ on V we have $\mathrm{I}(+v) = \mathrm{F}(+v)$ and $\mathrm{I}(-v) = \mathrm{F}(-v)$ for $v \in V$.)

Lemma 4.8: *Let $\mathcal{I}(\mathcal{P})$ be the set of all the ideals of a signed poset $\mathcal{P}$ on V. Then $\mathcal{I}(\mathcal{P})$ is a simple and spanning $\{\sqcup,\sqcap\}$-closed family on V.*

(Proof) We know that $\mathcal{I}(\mathcal{P})$ is $\{\sqcup,\sqcap\}$-closed family (see Lemma 3.18). Hence, it suffices to prove that $\mathcal{I}(\mathcal{P})$ is simple and spanning.

First, we show that $\mathcal{I}(\mathcal{P})$ is spanning. By the definition of the signed poset $\mathcal{P} = (V, A; \partial)$, for any $v \in V$ we have $2v \notin \partial A$ or $-2v \notin \partial A$. Therefore, there exists a nonnull ideal $\mathrm{I}(+v)$ or $\mathrm{I}(-v)$ for any $v \in V$, and hence, $\mathcal{I}(\mathcal{P})$ is spanning since $\mathcal{I}(\mathcal{P})$ is $\{\sqcup, \sqcap\}$-closed.

Next, we show that $\mathcal{I}(\mathcal{P})$ is simple. For any distinct $v, w \in V$, suppose $2v \notin \partial A$ without loss of generality. If $w \notin \mathrm{I}(+v)^+ \cup \mathrm{I}(+v)^-$, then we are done and if $w \in \mathrm{I}(+v)^+ \cup \mathrm{I}(+v)^-$, then $(\mathrm{I}(+v)^+ - \{v\}, \mathrm{I}(+v)^-)$ is a desired ideal that separates v and w. □

Lemma 4.9: *For two signed posets $\mathcal{P} = (V, A; \partial)$ and $\mathcal{P}' = (V, A'; \partial')$ on V, if $\partial A \neq \partial' A'$, then $\mathcal{I}(\mathcal{P}) \neq \mathcal{I}(\mathcal{P}')$.*

(Proof) If there exists some $v \in V$ such that $2v \in \partial A - \partial' A'$ (or $-2v \in \partial A - \partial' A'$), then the positive (or negative) principal ideal at v of $\mathcal{P}'$ is not contained in $\mathcal{I}(\mathcal{P})$. So, suppose that $\mathcal{P}$ and $\mathcal{P}'$ have the same set of selfloops. If there exist distinct $v, w \in V$ such that $v - w \in \partial A - \partial' A'$, then $2v \notin \partial' A'$ or $-2w \notin \partial' A'$. If $2v \notin \partial' A'$ (or $-2w \notin \partial' A'$), then for the positive (or negative) principal ideal W at v (or w) of $\mathcal{P}'$ we have $w \notin W^+$ (or $v \notin W^-$). Hence, W is not an ideal of $\mathcal{P}$. Other cases are treated similarly. □

From Theorem 4.7, Lemma 4.8 and Lemma 4.9 we have the main theorem.

Theorem 4.10: *There exists a one-to-one correspondence between the set of all the simple and spanning $\{\sqcup, \sqcap\}$-closed families $\mathcal{F} \subseteq 3^V$ on V and the set of all the signed posets on V such that each such $\mathcal{F}$ is the set of all the ideals of the corresponding signed poset $\mathcal{P}$. In fact, such a one-to-one correspondence is obtained by making each $\mathcal{F}$ correspond to $\mathcal{P}(\mathcal{F})$.* □

This is a set-theoretical version of the signed Birkhoff theorem of V. Reiner

4.2. General $\{\sqcup, \sqcap\}$-closed Families

In this section we discuss the representations of non-simple or non-spanning $\{\sqcup, \sqcap\}$-closed families.

Consider any $\{\sqcup, \sqcap\}$-closed family $\mathcal{F} \subseteq 3^V$. If a $\{\sqcup, \sqcap\}$-closed family $\mathcal{F}$ on V does not contain $(\emptyset, \emptyset)$, then for the minimum element (X_0, Y_0) of $\mathcal{F}$ define

$$\mathcal{F}' = \{(X - X_0, Y - Y_0) \mid (X, Y) \in \mathcal{F}\}. \tag{4.11}$$

Then, $\mathcal{F}'$ is a $\{\sqcup, \sqcap\}$-closed family on V with $(\emptyset, \emptyset) \in \mathcal{F}'$ and $\mathcal{F}'$ is isomorphic to $\mathcal{F}$. Hence, we can assume $(\emptyset, \emptyset) \in \mathcal{F}$.

Now, consider any $\{\sqcup, \sqcap\}$-closed family $\mathcal{F} \subseteq 3^V$ with $(\emptyset, \emptyset) \in \mathcal{F}$. Define an equivalence relation $\sim$ on $\mathrm{Supp}(\mathcal{F})$ as follows. For any $v, w \in \mathrm{Supp}(\mathcal{F})$ we have $v \sim w$ if and only if for each $(X, Y) \in \mathcal{F}$ either $v, w \in X \cup Y$ or $v, w \notin X \cup Y$. The equivalence classes associated with the equivalence relation $\sim$ give a partition $\Pi(\mathcal{F})$ of $\mathrm{Supp}(\mathcal{F})$. By the definition of the equivalence relation we see that each component $W \in \Pi(\mathcal{F})$ is divided into two sets W_1 and W_2 (either but not both possibly empty) such that for each $(X, Y) \in \mathcal{F}$ with $W \subseteq X \cup Y$ we have either

(1) $W_1 \subseteq X$ and $W_2 \subseteq Y$, or

(2) $W_1 \subseteq Y$ and $W_2 \subseteq X$.

Therefore, we should consider $\Pi(\mathcal{F})$ as a *double* partition, where each component $W \in \Pi(\mathcal{F})$ is further partitioned into two sets W_1 and W_2.

Summarizing, we have the following lemma.

Lemma 4.11: *For any $\{\sqcup, \sqcap\}$-closed family $\mathcal{F} \subseteq 3^V$ with $(\emptyset, \emptyset) \in \mathcal{F}$ there uniquely exists a partition $\Pi(\mathcal{F})$ of* $\mathrm{Supp}(\mathcal{F})$ *such that two elements $v, w \in$* $\mathrm{Supp}(\mathcal{F})$ *belong to the same component of* $\Pi(\mathcal{F})$ *if and only if there is no $(X, Y) \in \mathcal{F}$ that separates v and w. Furthermore, each component $W \in \Pi(\mathcal{F})$ is uniquely decomposed into two parts W_1 and W_2 such that for any $(X, Y) \in \mathcal{F}$ with $(W_1 \cup W_2) \cap (X \cup Y) \neq \emptyset$ we have either $W_1 \subseteq X$ and $W_2 \subseteq Y$, or $W_1 \subseteq Y$ and $W_2 \subseteq X$. Here, either W_1 or W_2 (but not both) may be empty.* □

Recall that for any family $\mathcal{F} \subseteq 3^V$ and any subset $U \subseteq V$ the reflection $\mathcal{F}\!:\!U$ of $\mathcal{F}$ by U is define as

$$\mathcal{F}\!:\!U = \{(X, Y)\!:\!U \mid (X, Y) \in \mathcal{F}\}, \tag{4.12}$$

where $(X, Y)\!:\!U$ is defined by

$$(X, Y)\!:\!U = ((X - U) \cup (Y \cap U), (Y - U) \cup (X \cap U)) \tag{4.13}$$

for each $(X, Y) \in \mathcal{F}$.

If either W_1 or W_2 is empty for each $W \in \Pi(\mathcal{F})$ in Lemma 4.11, we call $\mathcal{F}$ *homogeneous.* If $\mathcal{F}$ is not homogeneous, defining $U = \bigcup\{W_2 \mid W \in \Pi(\mathcal{F})\}$, the reflection $\mathcal{F}\!:\!U$ is homogeneous and is called a *homogenization* of $\mathcal{F}$. Define

$$\hat{\mathcal{F}} = \{(\hat{X}, \hat{Y}) \mid (X, Y) \in \mathcal{F}\!:\!U\}. \tag{4.14}$$

where for each $(X, Y) \in \mathcal{F}\!:\!U$

$$(\hat{X}, \hat{Y}) = (\{W \mid W \in \Pi(\mathcal{F}), W \subseteq X\}, \{W \mid W \in \Pi(\mathcal{F}), W \subseteq Y\}). \tag{4.15}$$

Then, $\hat{\mathcal{F}}$ is a simple and spanning $\{\sqcup, \sqcap\}$-closed family on $\Pi(\mathcal{F})$, and hence, is represented by a signed poset due to Theorem 4.5. We call $\hat{\mathcal{F}}$ *the simplification* of $\mathcal{F}$. For the homogenization of $\mathcal{F}$ we can choose either W_1 or W_2 for each component $W \in \Pi(\mathcal{F})$. The signed poset on $\Pi(\mathcal{F})$ is unique up to homogenization and is called the *signed poset derived from* $\mathcal{F}$.

Simplification of bisubmodular systems

We call a bisubmodular system $(\mathcal{F}, f)$ on V *simple* if the $\{\sqcup, \sqcap\}$-closed family $\mathcal{F}$ is simple.

Suppose that $(\mathcal{F}, f)$ is a non-simple bisubmodular system on V. Let $\mathcal{F}\!:\!U$ and $\hat{\mathcal{F}}$ be, respectively, a homogenization and the corresponding simplification of $\mathcal{F}$. Define $\hat{f}\colon \hat{\mathcal{F}} \to \mathbf{R}$ by

$$\hat{f}(\hat{X}, \hat{Y}) = f(X, Y) \quad ((X, Y) \in \mathcal{F}\!:\!U). \tag{4.16}$$

Then, we have a simple bisubmodular system $(\hat{\mathcal{F}}, \hat{f})$ on $\Pi(\mathcal{F})$. We call $(\hat{\mathcal{F}}, \hat{f})$ the *simplification* of the bisubmodular system $(\mathcal{F}, f)$.

Proper bisubmodular systems ([8])

A bisubmodular system $(\mathcal{F}, f)$ on V is called *proper* if we have $\mathcal{F} = \mathcal{I}(G)$ for some balanced bidirected graph $G = (V, A; \partial)$.

Theorem 4.12: *A bisubmodular system $(\mathcal{F}, f)$ on V is proper if and only if we have $(S, T), (T, S) \in \mathcal{F}$ for some orthant $(S, T) \in 3^V$.*

(Proof) The "only if" part follows from Lemma 2.8. The "if" part: Suppose that $\mathcal{F} \subseteq 3^V$ is $\{\sqcup, \sqcap\}$-closed and $(S, T), (T, S) \in \mathcal{F}$ for some orthant $(S, T) \in 3^V$. Then by Lemma 2.8 the signed poset $\mathcal{P}(\mathcal{F})$ derived from $\mathcal{F}$ must be balanced. We can extend the signed poset $\mathcal{P}(\mathcal{F})$ to a balanced bidirected graph $G = (V, A; \partial)$ such that $\mathcal{F} = \mathcal{I}(G)$. $\square$

Chapter 5

Structures of Bisubmodular Polyhedra

Based on results in Chapters 3 and 4, we shall examine structures of bisubmodular polyhedra in terms of signed posets and, more generally, exchangeability graphs. A different approach to the same subject was also presented by R. Guha, S. N. Kabadi and P. Sharma in [40].

In Section 5.2 we show characterizations of pointedness and boundedness of bisubmodular polyhedra. A characterization of extreme points is given in Section 5.3, where an algorithm for discerning whether a given point is an extreme point is provided. The algorithm also determines the signed poset associated with the given point if it is an extreme point. In Section 5.4 we consider linear optimization problems over a bisubmodular polyhedron and characterize optimal solutions in terms of exchangeability graphs. Characterizations of faces and their dimensions are also given in Section 5.5. Section 5.6 reveals the adjacency relation of extreme points by means of the Hasse diagrams of the associated signed posets. In Section 5.7 we investigate the connectivity and the decomposition of a bisubmodular system into its connected components.

This chapter is based on the joint work [5] with S. Fujishige.

5.1. Exchangeability Graphs and $\{\sqcup, \sqcap\}$-closed Families

For any $x \in \mathrm{P}_*(f)$ and any $v \in V$, if we have

$$\forall \alpha > 0\colon x + \alpha\chi_v \notin \mathrm{P}_*(f), \tag{5.1}$$

we say x is *positively saturated at* v, where χ_v is a unit vector in $\{0,1\}^V$ defined by $\chi_v(v) = 1$ and $\chi_v(w) = 0$ for $w \in V - \{v\}$. Similarly, we say x is *negatively saturated at* v if

$$\forall \alpha > 0\colon x - \alpha\chi_v \notin \mathrm{P}_*(f). \tag{5.2}$$

Denote by $\mathrm{sat}^{(+)}(x)$ (or $\mathrm{sat}^{(-)}(x)$) the set of elements of V at which x is positively (or negatively) saturated. Note that we may have $\mathrm{sat}^{(+)}(x) \cap \mathrm{sat}^{(-)}(x) \neq \emptyset$. We call

$\mathrm{sat}^{(+)}$ and $\mathrm{sat}^{(-)}$ the *signed saturation functions* ([37]), which generalize the saturation function for polymatroids and submodular systems (see [36]).

For $x \in \mathrm{P}_*(f)$, define $\mathcal{F}(x) \subseteq \mathcal{F}$ by

$$\mathcal{F}(x) = \{(X, Y) \mid (X, Y) \in \mathcal{F}, x(X, Y) = f(X, Y)\}. \tag{5.3}$$

We have the following (see [52], [19] for the case when $\mathcal{F} = 3^V$).

Lemma 5.1: *$\mathcal{F}(x)$ is closed with respect to $\sqcup$ and $\sqcap$.*
(Proof) Define $f' \colon \mathcal{F} \to \mathbf{R}$ by $f'(X, Y) = f(X, Y) - x(X, Y)$ $((X, Y) \in \mathcal{F})$. Then, f' is a bisubmodular function. Since f' is nonnegative and $f'(\emptyset, \emptyset) = 0$, $\mathcal{F}(x)$ is exactly the set of minimizers of f'. The present lemma follows from Lemma 2.2. □

Note that we have $v \in \mathrm{sat}^{(+)}(x)$ (or $v \in \mathrm{sat}^{(-)}(x)$) if and only if there exists some $(X, Y) \in \mathcal{F}(x)$ such that $v \in X$ (or $v \in Y$). Therefore, for any $v \in \mathrm{sat}^{(+)}(x)$ define

$$\mathrm{dep}(x, +v) = \sqcap\{(X, Y) \mid v \in X, (X, Y) \in \mathcal{F}(x)\}, \tag{5.4}$$

and for any $v \in \mathrm{sat}^{(-)}(x)$ define

$$\mathrm{dep}(x, -v) = \sqcap\{(X, Y) \mid v \in Y, (X, Y) \in \mathcal{F}(x)\}. \tag{5.5}$$

We call dep the *signed dependence function* ([37]), which generalizes the dependence function for polymatroids and submodular systems (see [36]). Note that $\mathrm{dep}(x, +v)$ ($\mathrm{dep}(x, -v)$) is a unique minimal element of $\mathcal{F}(x)$ whose positive (negative) part contains v.

For convenience, we also define $\mathrm{dep}(x, +v) = (\emptyset, \emptyset)$ for $v \in V - \mathrm{sat}^{(+)}(x)$ and $\mathrm{dep}(x, -v) = (\emptyset, \emptyset)$ for $v \in V - \mathrm{sat}^{(-)}(x)$.

We can easily see that for any $v \in \mathrm{sat}^{(+)}(x)$,

$$\mathrm{dep}(x, +v)^+ = \{w \mid w \in V, \exists \alpha > 0 \colon x + \alpha(\chi_v - \chi_w) \in \mathrm{P}_*(f)\}, \tag{5.6}$$

$$\mathrm{dep}(x, +v)^- = \{w \mid w \in V, \exists \alpha > 0 \colon x + \alpha(\chi_v + \chi_w) \in \mathrm{P}_*(f)\}. \tag{5.7}$$

Similarly, for any $v \in \mathrm{sat}^{(-)}(x)$,

$$\mathrm{dep}(x, -v)^+ = \{w \mid w \in V, \exists \alpha > 0 \colon x + \alpha(-\chi_v - \chi_w) \in \mathrm{P}_*(f)\}, \tag{5.8}$$

$$\mathrm{dep}(x, -v)^- = \{w \mid w \in V, \exists \alpha > 0 \colon x + \alpha(-\chi_v + \chi_w) \in \mathrm{P}_*(f)\}. \tag{5.9}$$

Define a bidirected graph $G(\mathcal{F}(x)) = (V, A(x); \partial)$ associated with $\mathcal{F}(x)$ as follows.

(1) For each $v \in V$,

(1a) there is a selfloop a at v with $\partial a = 2v$ if and only if $v \in V - \mathrm{sat}^{(+)}(x)$,

(1b) there is a selfloop a at v with $\partial a = -2v$ if and only if $v \in V - \mathrm{sat}^{(-)}(x)$.

(2) For each distinct $v, w \in V$,

(2a) there is an arc a with $\partial a = v - w$ if and only if $w \in \mathrm{dep}(x, +v)^+$ or $v \in \mathrm{dep}(x, -w)^-$,

(2b) there is an arc a with $\partial a = v + w$ if and only if $w \in \mathrm{dep}(x, +v)^-$ or $v \in \mathrm{dep}(x, +w)^-$,

(2c) there is an arc a with $\partial a = -v - w$ if and only if $w \in \mathrm{dep}(x, -v)^+$ or $v \in \mathrm{dep}(x, -w)^+$.

We call $G(\mathcal{F}(x))$ *the exchangeability (bidirected) graph* associated with $x \in \mathrm{P}_*(f)$. The collection of all the ideals of the exchangeability graph $G(\mathcal{F}(x))$ is $\mathcal{F}(x)$ as shown below.

Lemma 5.2: *Suppose that (X, Y) is an ideal of the exchangeability graph $G(\mathcal{F}(x)) = (V, A(x); \partial)$. Then we have for each $v \in X$*

$$v \in \mathrm{sat}^{(+)}(x), \quad \mathrm{dep}(x, +v) \sqsubseteq (X, Y) \tag{5.10}$$

and for each $v \in Y$

$$v \in \mathrm{sat}^{(-)}(x), \quad \mathrm{dep}(x, -v) \sqsubseteq (X, Y). \tag{5.11}$$

(Proof) This lemma easily follows from the definitions of ideal and exchangeability graph $G(\mathcal{F}(x))$ □

Theorem 5.3: *The set of all the ideals of $G(\mathcal{F}(x)) = (V, A(x); \partial)$ coincides with $\mathcal{F}(x)$.*
(Proof) Suppose that $(X, Y) \in 3^V$ is an ideal of $G(\mathcal{F}(x)) = (V, A(x); \partial)$. Then, we have from Lemma 5.2 that

$$(X, Y) = (\sqcup_{v \in X} \mathrm{dep}(x, +v)) \sqcup (\sqcup_{v \in Y} \mathrm{dep}(x, -v)). \tag{5.12}$$

It follows from Lemma 5.1 that $(X, Y) \in \mathcal{F}(x)$.

Conversely, suppose $(X, Y) \in \mathcal{F}(x)$. Then, by the definition of signed saturation function, we have

$$X \subseteq \mathrm{sat}^{(+)}(x), \tag{5.13}$$

$$Y \subseteq \mathrm{sat}^{(-)}(x). \tag{5.14}$$

Also, by the definition of signed dependence function, we have

$$\mathrm{dep}(x, +v) \sqsubseteq (X, Y) \qquad (v \in X), \tag{5.15}$$

$$\mathrm{dep}(x, -v) \sqsubseteq (X, Y) \qquad (v \in Y). \tag{5.16}$$

It follows from (5.13)∼(5.16) that (X, Y) is an ideal of $G(\mathcal{F}(x))$ (see the proof of Theorem 4.7). □

5.2. Pointedness and Boundedness

The *lineality space* of $\mathrm{P}_*(f)$ is the solution set of the following system of linear equations:

$$x(X,Y)=0 \quad ((X,Y)\in\mathcal{F}) \tag{5.17}$$

(see (1.19)). The bisubmodular polyhedron $\mathrm{P}_*(f)$ is pointed if and only if (5.17) has a unique solution $x=\mathbf{0}$. Recall that $\mathcal{F}$ is spanning due to the definition of bisubmodular system $(\mathcal{F},f)$.

Lemma 5.4: *A bisubmodular polyhedron* $\mathrm{P}_*(f)$ *is pointed if and only if* $\mathcal{F}$ *is simple (and spanning).*
(Proof) Note that $\mathcal{F}$ is spanning by definition. We can easily see that (5.17) is equivalent to

$$x(W_1)-x(W_2)=0 \quad (W\in\Pi(\mathcal{F})), \tag{5.18}$$

where $\Pi(\mathcal{F})$ is the partition of $\mathrm{Supp}(\mathcal{F})$ and $\{W_1,W_2\}$ is the bipartition of $W\in\Pi(\mathcal{F})$ that appeared in Lemma 4.11. $\mathrm{P}_*(f)$ is pointed if and only if (5.18) has a unique solution $x=\mathbf{0}$. This is the case if and only if $\mathrm{Supp}(\mathcal{F})=V$ and $|W|=1$ for each $W\in\Pi(\mathcal{F})$, i.e., $\mathcal{F}$ is spanning and simple. □

It should be noted that (5.18) gives an efficient representation of the lineality space of $\mathrm{P}_*(f)$ and that the dimension of the lineality space is equal to $|V|-|\Pi(\mathcal{F})|$.

Now, suppose that the underlying family $\mathcal{F}$ is simple and spanning. Therefore, $\mathcal{F}$ can be represented by a signed poset $\mathcal{P}(\mathcal{F})=(V,A;\partial)$ uniquely defined from $\mathcal{F}$ and $\mathcal{F}$ is the collection of all the ideals of $\mathcal{P}(\mathcal{F})$ (see Theorem 4.5). The *characteristic cone* $\mathrm{C}(\mathcal{F})$ of the bisubmodular polyhedron $\mathrm{P}_*(f)$ is given by

$$\mathrm{C}(\mathcal{F})=\{x\mid x\in\mathbf{R}^V,\forall(X,Y)\in\mathcal{F}\colon x(X,Y)\le 0\} \tag{5.19}$$

(see (1.18)). $\mathrm{P}_*(f)$ is bounded if and only if the system of inequalities appearing in the right-hand side of (5.19) has a unique solution $x=\mathbf{0}$. If the arc set A of the signed poset $\mathcal{P}(\mathcal{F})=(V,A;\partial)$ is nonempty, then for any arc $a\in A$ the vector $x=\partial a$ (considered as a vector in $\mathbf{R}^V$) satisfies the inequalities in (5.19). Hence, $\mathrm{P}_*(f)$ is not bounded. On the other hand, if $A=\emptyset$, then we have $\mathcal{F}=3^V$, so that $\mathrm{P}_*(f)$ is bounded.

Consequently, we have

Lemma 5.5: *A bisubmodular polyhedron* $\mathrm{P}_*(f)$ *is bounded if and only if we have* $\mathcal{F}=3^V$. □

Consider a capacity function c on the arc set A of the signed poset $\mathcal{P}(\mathcal{F})$ such that $c(a) = +\infty$ for all $a \in A$. A feasible flow φ in the bidirected network $\mathcal{N} = (\mathcal{P}(\mathcal{F}), c)$ is a function $\varphi\colon A \to \mathbf{R}$ such that $0 \le \varphi(a) \le c(a)$ $(a \in A)$. Recall that the boundary $\partial\varphi$ of a feasible flow φ in $\mathcal{N}$ is the vector in $\mathbf{R}^V$ defined by

$$\partial\varphi = \sum\{\varphi(a)\partial a \mid a \in A\}, \tag{5.20}$$

where ∂a $(a \in A)$ are regarded as vectors in $\mathbf{R}^V$. It follows from Corollary 3.7 that the set of the boundaries of all the feasible flows in $\mathcal{N}$ is exactly the characteristic cone $\mathrm{C}(\mathcal{F})$ in (5.19).

From this we have

Theorem 5.6: *Suppose that $\mathcal{F}$ is simple and spanning and let $\mathcal{P}(\mathcal{F}) = (V, A; \partial)$ be the signed poset on V representing $\mathcal{F}$. Then, the characteristic cone $\mathrm{C}(\mathcal{F})$ of $\mathrm{P}_*(f)$ is generated by $\{\partial a \mid a \in A\}$. Moreover, extreme rays of $\mathrm{C}(\mathcal{F})$ are exactly given by ∂a for arcs a of the Hasse diagram of $\mathcal{P}(\mathcal{F})$.* □

The latter part of Theorem 5.6 easily follows from the definition of Hasse diagram.

5.3. Extreme Points and Signed Posets

Associated with a point $x \in \mathrm{P}_*(f)$ we have a $\{\sqcup, \sqcap\}$-closed family $\mathcal{F}(x)$. Note that $x \in \mathrm{P}_*(f)$ is an extreme point of $\mathrm{P}_*(f)$ only if $\mathcal{F}(x)$ spans V. Also, if $\mathcal{F}(x)$ spans V, then f restricted to $\mathcal{F}(x)$ defines a bisubmodular system on V and x is the unique extreme point of the associated bisubmodular polyhedron if it is pointed. Therefore, we have from Lemma 5.4

Theorem 5.7: *A point $x \in \mathrm{P}_*(f)$ is an extreme point of $\mathrm{P}_*(f)$ if and only if $\mathcal{F}(x)$ is simple and spanning.* □

It follows from Theorem 5.7 that for each extreme point $x \in \mathrm{P}_*(f)$ we have a signed poset $\mathcal{P}(\mathcal{F}(x))$ representing the simple and spanning $\mathcal{F}(x)$. It follows from Theorem 5.3 that for each extreme point the signed poset $\mathcal{P}(\mathcal{F}(x))$ is essentially the same as the exchangeability graph $G(\mathcal{F}(x))$; they have the same set of ideals.

The signed poset $\mathcal{P}(\mathcal{F}(x))$ can be constructed if we are given $\mathrm{dep}(x, \pm v)$ $(v \in V)$. We can obtain these $\mathrm{dep}(x, \pm v)$ $(v \in V)$ by adapting an algorithm of Bixby, Cunningham and Topkis [15] for polymatroids to bisubmodular systems (also see [36, p. 62]). The following algorithm consists of two parts, Algorithm I and Algorithm II. Algorithm I discerns whether a given $x \in \mathbf{R}^V$ is an extreme point of $\mathrm{P}_*(f)$ and, if x is an extreme point, Algorithm II finds all $\mathrm{dep}(x, \pm v)$ $(v \in V)$, using the output of Algorithm I. It should be noted that we do not need Algorithm II for polymatroids or submodular systems.

Suppose we are given a vector $x \in \mathbf{R}^V$. We also define $\mathcal{F}(x)$ by (5.3) for x not necessarily in $\mathrm{P}_*(f)$.

Algorithm I

Step 1: Put $S \leftarrow (\emptyset, \emptyset)$.

Step 2: For each $i = 1, 2, \cdots, |V|$ do the following (2-1) and (2-2).

(2-1) If there exists no element $v \in V - (S^+ \cup S^-)$ such that
 (i) $(S^+ \cup \{v\}, S^-) \in \mathcal{F}(x)$ or
 (ii) $(S^+, S^- \cup \{v\}) \in \mathcal{F}(x)$
then stop (x is not an extreme point of $\mathrm{P}_*(f)$).
Otherwise let v be an element of $V - (S^+ \cup S^-)$ that satisfies (i) or (ii) and put $v_i \leftarrow v$.
If (i) is satisfied by $v = v_i$, then put $S \leftarrow (S^+ \cup \{v_i\}, S^-)$ and $\sigma(v_i) \leftarrow +1$.
Otherwise put $S \leftarrow (S^+, S^- \cup \{v_i\})$ and $\sigma(v_i) \leftarrow -1$.

(2-2) Put $T \leftarrow S$ and for each $j = 1, 2, \cdots, i-1$ do the following $(*)$:
$(*)$ If $(T^+ - \{v_{i-j}\}, T^-) \in \mathcal{F}(x)$ when $\sigma(v_{i-j}) = +1$,
 or if $(T^+, T^- - \{v_{i-j}\}) \in \mathcal{F}(x)$ when $\sigma(v_{i-j}) = -1$,
 then put $T \leftarrow (T^+ - \{v_{i-j}\}, T^-)$ or $T \leftarrow (T^+, T^- - \{v_{i-j}\})$ according as
 $\sigma(v_{i-j}) = +1$ or $\sigma(v_{i-j}) = -1$.
If $\sigma(v_i) = +1$, then put $\mathrm{dep}(x, +v_i) \leftarrow T$.
Otherwise put $\mathrm{dep}(x, -v_i) \leftarrow T$.

(End)

When Algorithm I terminates with an orthant $S = (S^+, S^-)$, we move to Algorithm II given as follows. Here, we assume $S^+ = \emptyset$ to simplify the description of the algorithm. If necessary, consider the reflections $(\mathcal{F}, f) : S^+$ and $x : S^+$ of the inputs of Algorithm I and the reflections $\mathrm{dep}(x, \pm v_i) : S^+$ $(i = 1, 2, \cdots, |V|)$ of the outputs.

Now, we have the orthant $(\emptyset, V) \in \mathcal{F}(x)$ and $\mathrm{dep}(x, -v_i)$ $(i = 1, 2, \cdots, |V|)$. Define a binary relation $\preceq$ on V by $u \preceq v$ if and only if $v \in \mathrm{dep}(x, -u)^-$. We can easily see that the binary relation $\preceq$ is a partial order and it gives an ordinary poset $\mathcal{P}_0 = (V, \preceq)$. For each $v \in V$ denote by $D(v)$ the principal order ideal of v in $\mathcal{P}_0$, i.e.,

$$D(v) = \{u \mid u \in V, u \preceq v\}. \tag{5.21}$$

We can obtain all $D(v)$ $(v \in V)$ in $\mathrm{O}(|V|^2)$ time, which are used in Algorithm II.

Algorithm II

Step 1: Put $W \leftarrow V$.

Step 2: For each $v \in V$, if $(D(v), \emptyset) \in \mathcal{F}(x)$, then put $\mathrm{dep}(x, +v) \leftarrow (D(v), \emptyset)$ and $W \leftarrow W - \{v\}$.
Step 3: For each $v \in W$, if $(D(v), W - D(v)) \notin \mathcal{F}(x)$, then put $\mathrm{dep}(x, +v) \leftarrow (\emptyset, \emptyset)$, otherwise do the following $(**)$.
$(**)$ Let $u_1, u_2, \cdots, u_k$ be the elements of $W - D(v)$ arranged in the topological order in $\mathcal{P}_0$ restricted on $W - D(v)$ (i.e., $u_i \prec u_j$ implies $i < j$).
Put $T \leftarrow (D(v), W - D(v))$.
For each $i = 1, 2, \cdots, k$, if $(T^+, T^- - \{u_i\}) \in \mathcal{F}(x)$, then put $T \leftarrow (T^+, T^- - \{u_i\})$.
Put $\mathrm{dep}(x, +v) \leftarrow T$.
(End)

The validity of Algorithm I can be shown as follows. If x is an extreme point of $\mathrm{P}_*(f)$, then $\mathcal{F}(x)$ is simple and spanning due to Theorem 5.7, and hence, any maximal chain

$$\mathcal{C}\colon (\emptyset, \emptyset) = (S_0^+, S_0^-) \sqsubset (S_1^+, S_1^-) \sqsubset \cdots \sqsubset (S_n^+, S_n^-) \tag{5.22}$$

of signed subsets $(S_i^+, S_i^-) \in \mathcal{F}(x)$ $(i = 0, 1, \cdots, n)$ satisfies $|S_i^+ \cup S_i^-| = i$ $(i = 0, 1, \cdots, n)$ with $n = |V|$. Therefore, if x is an extreme point of $\mathrm{P}_*(f)$, a maximal chain of $\mathcal{F}(x)$ of length n is found by Step (2-1). Moreover, if Step (2-1) finds a maximal chain of $\mathcal{F}(x)$ of length n, then $\mathcal{F}(x)$ is simple and spanning, so that x is an extreme point of $\mathrm{P}_*(f)$. This proves the validity of Step (2-1).

Now, let us consider Step (2-2). At an execution of Step (2-2) when (i) of Step (2-1) is satisfied by $v = v_i$, we have $\mathrm{dep}(x, +v_i) \neq (\emptyset, \emptyset)$ and from Step (2-1) we have obtained a chain

$$(\emptyset, \emptyset) = (S_0^+, S_0^-) \sqsubset (S_1^+, S_1^-) \sqsubset \cdots \sqsubset (S_i^+, S_i^-). \tag{5.23}$$

Because of the definition of $\mathrm{dep}(x, +v_i)$ we have

$$\mathrm{dep}(x, +v_i) \sqsubseteq (S_i^+, S_i^-). \tag{5.24}$$

It follows from (5.23) and (5.24) that the distinct members of $\mathrm{dep}(x, +v_i) \sqcup (S_j^+, S_j^-)$ $(j = 0, 1, \cdots, i)$ form a maximal chain from $\mathrm{dep}(x, +v_i)$ to (S_i^+, S_i^-). Therefore, removing possible elements from (S_i^+, S_i^-) one by one, we reach $\mathrm{dep}(x, +v_i)$. This validates Step (2-2). The validity of Step (2-2) when (ii) is satisfied can be shown similarly.

We show the validity of Algorithm II. We assume that the orthant $(\emptyset, V)$ is obtained by Algorithm I. Under this assumption the signed poset $\mathcal{P}(\mathcal{F}(x))$ does not contain any arcs a of type $\partial a = -v - w$. Hence, in particular, $\mathcal{P}(\mathcal{F}(x))$ does not contain any negative selfloops. Therefore, if $w \in \mathrm{dep}(x, +v)^+$, we have $v \in \mathrm{dep}(x, -w)^-$ (see Lemma 4.3). This means that Algorithm I has already obtained all the arcs a of type $\partial a = v - w$ for $v, w \in V$ in $\mathcal{P}(\mathcal{F}(x))$ by means of $\mathrm{dep}(x, -v)$ $(v \in V)$. The subgraph of $\mathcal{P}(\mathcal{F}(x))$

induced by the set of arcs a of type $\partial a = v - w$ ($v, w \in V$) is isomorphicly expressed by the poset $\mathcal{P}_0 = (V, \preceq)$ defined above. For each $v \in V$, if there is not a positive selfloop incident to v in $\mathcal{P}(\mathcal{F}(x))$, we have $(D(v), V - D(v)) \in \mathcal{F}(x)$ since there is no arc a such that $\partial a = u + w$ for $u, w \in D(v)$ (by the assumption) or $\partial a = u - w$ for $u \in D(v)$ and $w \in V - D(v)$ (by the definition of $D(v)$). Also, we can easily see that there is no $(X, Y) \in \mathcal{F}(x)$ such that $v \in X \subset D(v)$ (strict inclusion). Therefore, we only have to delete from $(D(v), V - D(v))$ as many elements in $V - D(v)$ as possible to obtain $\mathrm{dep}(x, +v)$. If there is no arc a of type $\partial a = u + w$ in $\mathcal{P}(\mathcal{F}(x))$ that is incident to any vertex of $D(v)$, we have $(D(v), \emptyset) \in \mathcal{F}(x)$ and hence $\mathrm{dep}(x, +v) = (D(v), \emptyset)$. This case is treated by Step 2 of Algorithm II. Note also that if $(D(v), \emptyset) \notin \mathcal{F}(x)$, then there is an arc a in $\mathcal{P}(\mathcal{F}(x))$ such that $\partial a = u + w$ and u or w belongs to $D(v)$. The set $W \subseteq V$ obtained after Step 2 is the set of the vertices to which some arc a in $\mathcal{P}(\mathcal{F}(x))$ of type $\partial a = u + w$ is incident. It follows that for each $v \in W$, if there is no positive selfloop incident to v, then we have $(D(v), W - D(v)) \in \mathcal{F}(x)$. We see that any maximal chain

$$\mathrm{dep}(x, +v)^- = T_0^- \subset T_1^- \subset \cdots \subset T_l^- = W - D(v) \tag{5.25}$$

of upper order ideals of $\mathcal{P}_0 = (V, \preceq)$ from $\mathrm{dep}(x, +v)^-$ to $W - D(v)$ gives a maximal chain

$$\mathrm{dep}(x, +v) = (T_0^+, T_0^-) \sqsubset (T_0^+, T_1^-) \cdots \sqsubset (T_0^+, T_l^-) = (D(v), W - D(v)) \tag{5.26}$$

of ideals of $\mathcal{P}(\mathcal{F}(x))$ from $\mathrm{dep}(x, +v)$ to $(D(v), W - D(v))$. Also, note that a maximal chain in (5.25) is formed by different T^-'s appearing in $(**)$ of Step 3 since we have $(T^+, T^- - \{u_i\}) \in \mathcal{F}(x)$ (at the iteration for i) if and only if $u_i \notin \mathrm{dep}(x, +v)^-$, due to the ordering of u_i ($i = 1, 2, \cdots, k$). Therefore, $\mathrm{dep}(x, +v)$ for each $v \in W$ is obtained by Step 3. It should be noted that a topological ordering of elements of $W - D(v)$ required in $(**)$ of Step 3 can be obtained in $\mathrm{O}(|V|)$ time for each $v \in W$ if we have once obtained a topological ordering of W, which requires $\mathrm{O}(|V|^2)$ time.

The total running time of Algorithms I and II is $\mathrm{O}(|V|^2)$ if we assume an oracle for function evaluation of f, while the algorithm of Bixby, Cunningham and Topkis [15] also requires $\mathrm{O}(|V|^2)$ time for polymatroids.

For any $x \in \mathrm{P}_*(f)$ the *tangent cone* $\mathrm{TC}(f, x)$ of $\mathrm{P}_*(f)$ at x is given by

$$\mathrm{TC}(f, x) = \{x \mid x \in \mathbf{R}^V, \forall (X, Y) \in \mathcal{F}(x) \colon x(X, Y) \leq 0\} \tag{5.27}$$

(see (1.20)). Therefore, from Theorems 5.6 and 5.7 we have the following

Theorem 5.8: *For any extreme point $x \in \mathrm{P}_*(f)$ the extreme rays of the tangent cone $\mathrm{TC}(f, x)$ are exactly given by the boundaries ∂a of arcs a of the Hasse diagram of the signed poset $\mathcal{P}(\mathcal{F}(x))$ representing $\mathcal{F}(x)$.* □

By an argument similar to the one around (5.20) we can show

Corollary 5.9: *For any point $x \in \mathrm{P}_*(f)$ the tangent cone $\mathrm{TC}(f, x)$ is generated by the boundaries of arcs of the exchangeability graph $G(\mathcal{F}(x))$.* □

5.4. Linear Optimization and an Optimality Condition

For a bisubmodular system $(\mathcal{F}, f)$ on V let us consider the following linear optimization problem:

$$\begin{aligned} (P_w) \quad & \text{Maximize} \textstyle\sum_{v \in V} w(v)x(v) \\ & \text{subject to } x \in \mathrm{P}_*(f), \end{aligned} \tag{5.28}$$

where $w \in \mathbf{R}^V$ is a weight vector. A greedy algorithm is given in [21], [52], [62] and [27] for the case $\mathcal{F} = 3^V$. We give a characterization of optimal solutions for this problem in terms of exchangeability graphs and examine the greedy algorithm in the case when $\mathcal{F} \neq 3^V$. For simplicity we assume that $\mathcal{F}$ is simple. Recall that $\mathcal{F}$ is spanning by the definition of $(\mathcal{F}, f)$.

For the signed poset $\mathcal{P}(\mathcal{F}) = (V, A; \partial)$ corresponding to $\mathcal{F}$ define

$$\mathrm{C}^*(\mathcal{F}) = \{z \mid z \in \mathbf{R}^V, \forall a \in A \colon \langle \partial a, z \rangle \leq 0\}. \tag{5.29}$$

We see from (5.19) and Theorem 5.6 that $\mathrm{C}^*(\mathcal{F})$ is the dual cone of the characteristic cone $\mathrm{C}(\mathcal{F})$ (see also Section 1.3). Therefore, we have the following

Lemma 5.10: *Problem (P_w) has a finite optimal solution if and only if $w \in \mathrm{C}^*(\mathcal{F})$.* □

Note that we can easily check whether $w \in \mathrm{C}^*(\mathcal{F})$ holds, using the signed poset $\mathcal{P}(\mathcal{F})$. It should also be noted that the cone $\mathrm{C}^*(\mathcal{F})$ is generated by $\chi_{(X,Y)}$ $((X, Y) \in \mathcal{F})$. This fact can also be derived from the argument given in the proof of Theorem 7.3.

For any $x \in \mathrm{P}_*(f)$ we have the exchangeability graph $G(\mathcal{F}(x)) = (V, A(x); \partial)$ associated with x. Define a cone

$$\mathrm{C}^*(\mathcal{F}(x)) = \{z \mid z \in \mathbf{R}^V, \forall a \in A(x) \colon \langle \partial a, z \rangle \leq 0\}. \tag{5.30}$$

It follows from Corollary 5.9 that $\mathrm{C}^*(\mathcal{F}(x))$ is the dual cone of the tangent cone $\mathrm{TC}(f, x)$ at x of $\mathrm{P}_*(f)$. The cone $\mathrm{C}^*(\mathcal{F}(x))$ is generated by $\chi_{(X,Y)}$ $((X, Y) \in \mathcal{F}(x))$.

Optimal solutions of Problem (P_w) are characterized by the following theorem.

Theorem 5.11: *A vector* $x \in \mathrm{P}_*(f)$ *is an optimal solution for Problem* (P_w) *if and only if* $w \in \mathrm{C}^*(\mathcal{F}(x))$.
(Proof) Since the cone $\mathrm{C}^*(\mathcal{F}(x))$ is the dual cone of the tangent cone $\mathrm{TC}(f, x)$ at x of $\mathrm{P}_*(f)$, the present theorem follows from Theorem 1.8. □

Suppose that $w \in \mathrm{C}^*(\mathcal{F})$. Let the distinct positive values of $|w(v)|$ $(v \in V)$ be given by

$$w_1 > w_2 > \cdots > w_p \ (> 0). \tag{5.31}$$

Define

$$U_i = \{v \mid v \in V, w(v) \geq w_i\} \quad (i = 1, 2, \cdots, p), \tag{5.32}$$

$$W_i = \{v \mid v \in V, -w(v) \geq w_i\} \quad (i = 1, 2, \cdots, p). \tag{5.33}$$

Since $w \in \mathrm{C}^*(\mathcal{F})$, for each $i = 1, 2, \cdots, p$ we have $(U_i, W_i) \in \mathcal{F}$. Moreover, since

$$(U_1, W_1) \sqsubset (U_2, W_2) \sqsubset \cdots \sqsubset (U_p, W_p) \tag{5.34}$$

is a chain of $\mathcal{F}$, it can be extended to a maximal chain of $\mathcal{F}$ as

$$(\emptyset, \emptyset) = (\hat{U}_0, \hat{W}_0) \sqsubset (\hat{U}_1, \hat{W}_1) \sqsubset (\hat{U}_2, \hat{W}_2) \sqsubset \cdots \sqsubset (\hat{U}_n, \hat{W}_n), \tag{5.35}$$

where the length n of the chain is equal to $|V|$ since $\mathcal{F}$ is simple and spanning. Suppose that

$$\{v_i\} = (\hat{U}_i \cup \hat{W}_i) - (\hat{U}_{i-1} \cup \hat{W}_{i-1}) \quad (i = 1, 2, \cdots, n). \tag{5.36}$$

Then, defining

$$x(v_i) = \begin{cases} f(\hat{U}_i, \hat{W}_i) - f(\hat{U}_{i-1}, \hat{W}_{i-1}) & (\text{if } \{v_i\} = \hat{U}_i - \hat{U}_{i-1}) \\ f(\hat{U}_{i-1}, \hat{W}_{i-1}) - f(\hat{U}_i, \hat{W}_i) & (\text{if } \{v_i\} = \hat{W}_i - \hat{W}_{i-1}) \end{cases} \tag{5.37}$$

for each $i = 1, 2, \cdots, n$, we have

$$x(\hat{U}_i, \hat{W}_i) = f(\hat{U}_i, \hat{W}_i) \quad (i = 1, 2, \cdots, n). \tag{5.38}$$

We can easily see that $x \in \mathrm{P}_*(f)$ and $w \in \mathrm{C}^*(\mathcal{F}(x))$. Therefore, the vector x given by (5.37) is an optimal solution for Problem (P_w) due to Theorem 5.11. This gives the *greedy algorithm* for Problem (P_w), which works for $\mathcal{F}$ not necessarily equal to 3^V.

The following theorem was given by Chandrasekaran and Kabadi [21] for the case when $\mathcal{F} = 3^V$.

Theorem 5.12: *Let* $f : \mathcal{F} \to \mathbf{R}$ *be a function on a simple and spanning* $\{\sqcup, \sqcap\}$*-closed family* $\mathcal{F}$ *on* V *such that* $(\emptyset, \emptyset) \in \mathcal{F}$ *and* $f(\emptyset, \emptyset) = 0$. *Define*

$$\mathrm{P}_*(f) = \{x \mid x \in \mathbf{R}^V, \forall (X, Y) \in \mathcal{F} \colon x(X, Y) \le f(X, Y)\}. \tag{5.39}$$

Then, the greedy algorithm described above works for $\mathrm{P}_*(f)$ *defined by* (5.39) *if and only if* f *is bisubmodular.*

(Proof) It suffices to prove "only if" part. Suppose that the greedy algorithm works for $\mathrm{P}_*(f)$ defined by (5.39). Then, for any maximal chain

$$\mathcal{C} \colon (\emptyset, \emptyset) = (\hat{U}_0, \hat{W}_0) \sqsubset (\hat{U}_1, \hat{W}_1) \sqsubset (\hat{U}_2, \hat{W}_2) \sqsubset \cdots \sqsubset (\hat{U}_n, \hat{W}_n) \tag{5.40}$$

of $\mathcal{F}$ the vector $x \in \mathbf{R}^V$ defined by (5.37) is an element of $\mathrm{P}_*(f)$. Now for any $(X_1, Y_1), (X_2, Y_2) \in \mathcal{F}$ let $\mathcal{C}$ in (5.40) be a maximal chain containing both $(X_1, Y_1) \sqcup (X_2, Y_2)$ and $(X_1, Y_1) \sqcap (X_2, Y_2)$. It follows from the definition of x that

$$x((X_1, Y_1) \sqcup (X_2, Y_2)) = f((X_1, Y_1) \sqcup (X_2, Y_2)), \tag{5.41}$$

$$x((X_1, Y_1) \sqcap (X_2, Y_2)) = f((X_1, Y_1) \sqcap (X_2, Y_2)). \tag{5.42}$$

Also, since $x \in \mathrm{P}_*(f)$, we have

$$x(X_1, Y_1) \le f(X_1, Y_1), \ \ x(X_2, Y_2) \le f(X_2, Y_2). \tag{5.43}$$

Therefore, we have

$$\begin{aligned} f(X_1, Y_1) + f(X_2, Y_2) &\ge x(X_1, Y_1) + x(X_2, Y_2) \\ &= x((X_1, Y_1) \sqcup (X_2, Y_2)) + x((X_1, Y_1) \sqcap (X_2, Y_2)) \\ &= f((X_1, Y_1) \sqcup (X_2, Y_2)) + f((X_1, Y_1) \sqcap (X_2, Y_2)). \end{aligned} \tag{5.44}$$

$\square$

For the minimization of a separable convex objective function over the integral points of a bisubmodular polyhedron, the incremental algorithm ([6], [9]) and the decomposition algorithm [37] are valid.

5.5. Faces

We adapt the general technique developed in [36, Section 3.3.d] to bisubmodular polyhedra.

For any $\mathcal{G} \subseteq \mathcal{F}$ define

$$\begin{aligned} \mathrm{F}(\mathcal{G}) &= \{x \mid x \in \mathbf{R}^V, \ \forall (X,Y) \in \mathcal{G} \colon x(X,Y) = f(X,Y), \\ &\qquad \forall (X,Y) \in \mathcal{F} - \mathcal{G} \colon x(X,Y) \le f(X,Y)\}, \end{aligned} \tag{5.45}$$

$$\begin{aligned} \mathrm{F}^\circ(\mathcal{G}) &= \{x \mid x \in \mathbf{R}^V, \ \forall (X,Y) \in \mathcal{G} \colon x(X,Y) = f(X,Y), \\ &\qquad \forall (X,Y) \in \mathcal{F} - \mathcal{G} \colon x(X,Y) < f(X,Y)\}. \end{aligned} \tag{5.46}$$

Also, define

$$\mathbf{G} = \{\mathcal{G} \mid \mathcal{G} \text{ is a } \{\sqcup, \sqcap\}\text{-closed subfamily of } \mathcal{F} \text{ with } (\emptyset,\emptyset) \in \mathcal{G}, \mathrm{F}^\circ(\mathcal{G}) \neq \emptyset\}. \tag{5.47}$$

Lemma 5.13: *The collection* $\mathbf{G}$ *of subfamilies of* $\mathcal{F}$ *defined by* (5.47) *is given by*

$$\mathbf{G} = \{\mathcal{F}(x) \mid x \in \mathrm{P}_*(f)\}, \tag{5.48}$$

where for each $x \in \mathrm{P}_*(f)$ $\mathcal{F}(x)$ *is defined by* (5.3).
(Proof) If $\mathcal{G} \in \mathbf{G}$, then for any $x \in \mathrm{F}^\circ(\mathcal{G})$ we have $\mathcal{G} = \mathcal{F}(x)$ by the definition (5.46). Conversely, for any $x \in \mathrm{P}^*(f)$ $\mathcal{F}(x)$ is a $\{\sqcup, \sqcap\}$-closed subfamily of $\mathcal{F}$ with $(\emptyset,\emptyset) \in \mathcal{F}(x)$ and $x \in \mathrm{F}^\circ(\mathcal{F}(x))$. Hence, $\mathcal{F}(x) \in \mathbf{G}$. □

From Lemma 5.13 we have the following

Theorem 5.14: *The collection* $\mathbf{F}$ *of all the nonempty faces of* $\mathrm{P}_*(f)$ *is given by* $\mathbf{F} = \{\mathrm{F}(\mathcal{G}) \mid \mathcal{G} \in \mathbf{G}\}$. *Also,*

(i) *If* $\mathcal{G}_1, \mathcal{G}_2 \in \mathbf{G}$ *and* $\mathcal{G}_1 \neq \mathcal{G}_2$, *then* $\mathrm{F}(\mathcal{G}_1) \neq \mathrm{F}(\mathcal{G}_2)$.

(ii) *For any* $\mathcal{G}_1, \mathcal{G}_2 \in \mathbf{G}$, $\mathcal{G}_1 \subseteq \mathcal{G}_2$ *if and only if* $\mathrm{F}(\mathcal{G}_1) \supseteq \mathrm{F}(\mathcal{G}_2)$.

In other words, $\mathrm{F}(\cdot)$ *defined by* (5.45) *determines an anti-order isomorphism from* $\mathbf{G}$ *to* $\mathbf{F}$, *where* $\mathbf{G}$ *and* $\mathbf{F}$ *are considered as posets relative to set inclusion.* □

Also, the dimensions of faces are given as follows.

Theorem 5.15: *For any* $\mathcal{G} \in \mathbf{G}$ *we have*

$$\mathrm{dim}\mathrm{F}(\mathcal{G}) = |V| - |\Pi(\mathcal{G})|, \tag{5.49}$$

where $\mathrm{dim}\mathrm{F}(\mathcal{G})$ *is the dimension of* $\mathrm{F}(\mathcal{G})$ *and* $\Pi(\mathcal{G})$ *is the partition of* $\mathrm{Supp}(\mathcal{G})$ *defined as in Theorem* 4.11.
(Proof) For any $\mathcal{G} \in \mathbf{G}$ the dimension of the face $\mathrm{F}(\mathcal{G})$ is equal to the dimension of the linear subspace formed by the solution vectors of the following system of linear equations:

$$x(X,Y) = 0 \quad ((X,Y) \in \mathcal{G}), \tag{5.50}$$

which is equivalent to

$$x(W_1) - x(W_2) = 0 \quad (W \in \Pi(\mathcal{G})) \tag{5.51}$$

as in the proof of Theorem 5.4. The present theorem easily follows from this. □

It should be noted that $|\Pi(\mathcal{G})|$ is equal to the length of any maximal chain of $\mathcal{G}$. Theorem 5.7 also follows from Lemma 5.15.

From Lemma 5.15 we also have

Corollary 5.16: *For any $\mathcal{G} \in \mathbf{G}$ let*

$$\mathcal{C}\colon (\emptyset, \emptyset) = (U_0^+, U_0^-) \sqsubset (U_1^+, U_1^-) \sqsubset \cdots \sqsubset (U_k^+, U_k^-) \tag{5.52}$$

be a maximal chain of $\mathcal{G}$. Then we have

$$\mathrm{F}(\mathcal{G}) = \mathrm{F}(\mathcal{C}). \tag{5.53}$$

(Proof) It follows from Theorem 5.15 and (5.52) that the coefficient matrix of the system of linear equations

$$x(X, Y) = 0 \quad ((X, Y) \in \mathcal{G}) \tag{5.54}$$

and that of its subsystem

$$x(X, Y) = 0 \quad ((X, Y) \in \mathcal{C}) \tag{5.55}$$

have the same rank $k = |\Pi(\mathcal{G})|$, the length of the maximal chain $\mathcal{C}$. Therefore, (5.54) and (5.55) determine the same solution set. Relation (5.53) follows from this fact. □

Since f is bimodular on $\mathcal{G} \in \mathbf{G}$, Corollary 5.16 also follows from Lemma 6.1.

5.6. Adjacency of Extreme Points

From Theorem 5.8 the edges, one-dimensional faces, of $\mathrm{P}_*(f)$ incident to an extreme point $x \in \mathrm{P}_*(f)$ are characterized as follows. Denote by $\mathcal{H}(x)$ the Hasse diagram of the signed poset $\mathcal{P}(\mathcal{F}(x))$ (or equivalently, of the exchangeability graph $G(\mathcal{F}(x))$).

Theorem 5.17: *Let x be any extreme point of $\mathrm{P}_*(f)$. For any arc a of the Hasse diagram $\mathcal{H}(x)$ associated with x let $G'(x, a)$ be the bidirected graph obtained by adding to $G(\mathcal{F}(x))$ the arc $\bar{a}$ with its boundary $\partial\bar{a} = -\partial a$. Then, we have $\mathcal{I}(G'(x, a)) \in \mathbf{G}$ and $\mathrm{F}(\mathcal{I}(G'(x, a)))$ is an edge of $\mathrm{P}_*(f)$ incident to x. Conversely, every edge incident to x is given for some arc a of $\mathcal{H}(x)$ in this way.*

(Proof) Note that edges incident to an extreme point x correspond to extreme rays of the tangent cone $\mathrm{TC}(f, x)$ given in Theorem 5.8. Therefore, for any arc a of the Hasse diagram $\mathcal{H}(x)$ a point $y = x + \epsilon\partial a$ with a sufficiently small positive real ϵ lies on (the relative interior of) the edge corresponding to the arc a. It follows from Theorem 5.14 that $\mathcal{F}(y) \subset \mathcal{F}(x)$. Also, we can easily see that

$$\mathcal{F}(x) - \mathcal{F}(y) = \{(X, Y) \mid (X, Y) \in \mathcal{F}(x), \langle \partial a, (X, Y) \rangle < 0\}. \tag{5.56}$$

Adding to $G(\mathcal{F}(x))$ the arc $\bar{a}$ with $\partial \bar{a} = -\partial a$ removes from $\mathcal{F}(x)$ $(= \mathcal{I}(G(\mathcal{F}(x))))$ exactly the signed subsets in (5.56) and gives $\mathcal{F}(y) = \mathcal{I}(G'(x, a))$. This proves the present theorem. □

Now, we can give a characterization of the adjacency of extreme points.

Theorem 5.18: *Two distinct extreme points x_1 and x_2 of $\mathrm{P}_*(f)$ are adjacent if and only if there exist arcs a_1 of $\mathcal{H}(x_1)$ and a_2 of $\mathcal{H}(x_2)$ such that $G'^*(x_1, a_1) = G'^*(x_2, a_2)$, where $G'^*(x_1, a_1)$ and $G'^*(x_2, a_2)$ are the transitive closures of $G'(x_1, a_1)$ and $G'(x_2, a_2)$, respectively.*

(Proof) The present theorem easily follows from Theorem 5.17, since we have that $\mathcal{I}(G'(x_1, a_1)) = \mathcal{I}(G'(x_2, a_2))$ if and only if $G'^*(x_1, a_1) = G'^*(x_2, a_2)$. □

Suppose that x_1 and x_2 are adjacent extreme points of $\mathrm{P}_*(f)$. Then the arcs a_1 and a_2 in the above theorem satisfy $\partial a_1 = -\partial a_2$ and may be (both) nonselfloops or selfloops. If they are both nonselfloops, then x_1 and x_2 are extreme points of the base polyhedron in some common orthant (S, T), i.e., $\mathcal{F}(x_1)$ and $\mathcal{F}(x_2)$ have a common maximal element (S, T) with $S \cup T = V$. Also, $\mathcal{I}(G'(x_1, a_1))(= \mathcal{I}(G'(x_2, a_2)))$ is spanning and pre-simple. On the other hand, if a_1 and a_2 are both selfloops, then x_1 and x_2 are not extreme points of the base polyhedron in any common orthant but $\mathcal{F}(x_1)$ and $\mathcal{F}(x_2)$ have a common chain of length $n - 1$ that is a maximal chain of $\mathcal{I}(G'(x_1, a_1)) = \mathcal{I}(G'(x_2, a_2))$. In this case $\mathcal{I}(G'(x_1, a_1))(= \mathcal{I}(G'(x_2, a_2)))$ is pre-spanning and simple.

It should be noted that two extreme points x_1 and x_2 of $\mathrm{P}_*(f)$ are adjacent if and only if $x_2 - x_1$ is a positive scalar multiple of the boundary of an arc of the Hasse diagram $\mathcal{H}(x_1)$, due to Theorem 5.8. The latter condition of the equivalence can be checked in $\mathrm{O}(|V|^2)$ time by using the exchangeability graph $G(\mathcal{F}(x_1))$. Since $G(\mathcal{F}(x_1))$ can be computed in $\mathrm{O}(|V|^2)$ time using the algorithms given in Section 3.2, the total running time for testing the adjacency of given two extreme points is $\mathrm{O}(|V|^2)$, where we assume a function evaluation oracle for f.

5.7. Connectivity and Connected Components

We say that a bisubmodular system $(\mathcal{F}, f)$ on V is *connected* if there does not exist any nonempty proper subset W of V such that for each orthant $(S, T) \in \mathcal{F}$ we have

$$(S \cap W, T \cap W), (S - W, T - W) \in \mathcal{F} \tag{5.57}$$

and

$$f(S,T) = f(S \cap W, T \cap W) + f(S - W, T - W). \tag{5.58}$$

Note that the above condition for connectedness is equivalent to a weaker one that there exists no nonempty proper subset W of V such that for each not necessarily orthant $(X,Y) \in \mathcal{F}$ we have (5.57) and (5.58) with (S,T) being replaced by (X,Y). To see this equivalence, suppose that there exists a nonempty proper subset W of V such that for any orthant $(S,T) \in \mathcal{F}$ we have (5.57) and (5.58). Let $(X,Y) \in \mathcal{F}$. Choose an orthant $(\hat{S},\hat{T}) \in \mathcal{F}$ such that $(X,Y) \sqsubseteq (\hat{S},\hat{T})$. Then, we have

$$\begin{aligned} f(X,Y) + f(\hat{S},\hat{T}) &= f(X,Y) + f(\hat{S} \cap W, \hat{T} \cap W) + f(\hat{S} - W, \hat{T} - W) \\ &\geq f(X \cap W, Y \cap W) + f(X - W, Y - W) + f(\hat{S},\hat{T}) \\ &\geq f(X,Y) + f(\hat{S},\hat{T}). \end{aligned} \tag{5.59}$$

Hence, $(X \cap W, Y \cap W), (X - W, Y - W) \in \mathcal{F}$ and $f(X,Y) = f(X \cap W, Y \cap W) + f(X - W, Y - W)$, and the latter condition implies the former one. The converse implication is trivial.

Lemma 5.19: *Suppose that $(\mathcal{F}, f)$ is a connected bisubmodular system on V. Then, there is no $(X,Y) \in 3^V$ such that $(X,Y), (Y,X) \in \mathcal{F}$, $f(X,Y) + f(Y,X) = 0$ and $X \cup Y \neq \emptyset, V$.*

(Proof) Let $(\mathcal{F}, f)$ be a connected bisubmodular system on V. Suppose that for $(X,Y) \in 3^V$ we have $(X,Y), (Y,X) \in \mathcal{F}$ and $f(X,Y) + f(Y,X) = 0$. Then, for any $(Z,U) \in \mathcal{F}$ we have

$$\begin{aligned} 2f(Z,U) &= 2f(Z,U) + f(X,Y) + f(Y,X) \\ &\geq f((Z,U) \sqcup (X,Y)) + f((Z,U) \sqcap (X,Y)) \\ &\quad + f((Z,U) \sqcup (Y,X)) + f((Z,U) \sqcap (Y,X)) \\ &\geq f(Z \cap (X \cup Y), U \cap (X \cup Y)) + f(Z - (X \cup Y), U - (X \cup Y)) \\ &\quad + f(Z,U) \\ &\geq 2f(Z,U). \end{aligned} \tag{5.60}$$

Therefore, putting $W = X \cup Y$ we have $(Z \cap W, U \cap W), (Z - W, U - W) \in \mathcal{F}$ and

$$f(Z,U) = f(Z \cap W, U \cap W) + f(Z - W, U - W) \tag{5.61}$$

for any $(Z,U) \in \mathcal{F}$. Since $(\mathcal{F}, f)$ is connected, we must have $W = \emptyset$ or $W = V$. $\square$

It should be noted that if $(X,Y),(Y,X) \in \mathcal{F}$ and $f(X,Y)+f(Y,X)=0$, then for any $x \in \mathrm{P}_*(f)$ we have $x(X,Y)=f(X,Y)$ and $x(Y,X)=f(Y,X)$, i.e., $(X,Y),(Y,X) \in \mathcal{F}(x)$, and conversely, that if $(X,Y),(Y,X) \in \mathcal{F}(x)$ for some $x \in \mathrm{P}_*(f)$, then we have $(X,Y),(Y,X) \in \mathcal{F}$ and $f(X,Y)+f(Y,X)=0$, and hence $(X,Y),(Y,X) \in \mathcal{F}(y)$ for any $y \in \mathrm{P}_*(f)$.

We call a connected bisubmodular system $(\mathcal{F},f)$ on V *fully connected* if there is no $(X,Y) \in 3^V$ such that $(X,Y),(Y,X) \in \mathcal{F}$, $f(X,Y)+f(Y,X)=0$ and $(X,Y) \neq (\emptyset,\emptyset)$. It should be noted that for a fully connected bisubmodular system $(\mathcal{F},f)$ the associated polyhedron $\mathrm{P}_*(f)$ is full-dimensional.

Suppose that $(\mathcal{F},f)$ is connected but not fully connected. Then, define

$$\mathcal{F}_0 = \{(X,Y) \mid (X,Y),(Y,X) \in \mathcal{F}, f(X,Y)+f(Y,X)=0\}. \tag{5.62}$$

It should be noted that there exists an orthant $(S,T) \in \mathcal{F}_0$ due to Lemma 5.19 and the definition of fully connectedness. If there exist two distinct orthants $(S_i,T_i) \in \mathcal{F}_0$ $(i=1,2)$ such that $(S_1,T_1) \neq (T_2,S_2)$, then

$$\begin{aligned} 0 &= f(S_1,T_1)+f(T_1,S_1)+f(S_2,T_2)+f(T_2,S_2) \\ &\geq f((S_1,T_1)\sqcup(S_2,T_2))+f((S_1,T_1)\sqcap(S_2,T_2)) \\ &\quad +f((T_1,S_1)\sqcup(T_2,S_2))+f((T_1,S_1)\sqcap(T_2,S_2)) \\ &\geq 0. \end{aligned} \tag{5.63}$$

This implies $(S_1,T_1)\sqcup(S_2,T_2)$ $(=(S_1,T_1)\sqcap(S_2,T_2)) \in \mathcal{F}_0$, which contradicts Lemma 5.19. Consequently, there exists a unique pair of signed subsets $(S,T),(T,S) \in \mathcal{F}_0$ such that $S \cup T = V$. From this we have the following theorem. For any signed subset (X,Y) we call (Y,X) the *reversal* of (X,Y).

Theorem 5.20: *Let $(\mathcal{F},f)$ be a connected but not fully connected bisubmodular system on V. Then there exists a unique orthant (S,T) (unique up to reversal) such that*

$$\mathrm{P}_*(f) = \mathrm{B}_{(S,T)}(f), \tag{5.64}$$

where $\mathrm{B}_{(S,T)}(f)$ is the base polyhedron of $(\mathcal{F},f)$ in the orthant (S,T). □

From Theorem 5.20 and a connectivity result on base polyhedra (see [36]) we also have

Corollary 5.21: *A bisubmodular system $(\mathcal{F},f)$ on V is connected but not fully connected if and only if $\{(\emptyset,\emptyset),(S,T),(T,S)\} \in \mathbf{G}$ for an orthant (S,T).* □

If a subset W of V satisfies (5.57) and (5.58) for each orthant $(S,T) \in \mathcal{F}$, we call W a *separator* of $(\mathcal{F}, f)$. Note that V and $\emptyset$ are separators of $(\mathcal{F}, f)$.

Lemma 5.22: *Suppose that $(\mathcal{F}, f)$ is a bisubmodular system on V. The set $\mathcal{W}$ of the separators of $(\mathcal{F}, f)$ forms a Boolean lattice with respect to set union and intersection as the lattice operations.*

(Proof) Let $W_1, W_2 \in \mathcal{W}$. By the definition of separators, for any orthant $(S,T) \in \mathcal{F}$ we have (5.57) and (5.58) with W being replaced by W_i for each $i = 1, 2$. It follows that for any orthant $(S,T) \in \mathcal{F}$ we have

$$(S \cap (W_1 \cap W_2), T \cap (W_1 \cap W_2)),\ (S - (W_1 \cap W_2), T - (W_1 \cap W_2)) \in \mathcal{F}, \tag{5.65}$$
$$(S \cap (W_1 \cup W_2), T \cap (W_1 \cup W_2)),\ (S - (W_1 \cup W_2), T - (W_1 \cup W_2)) \in \mathcal{F} \tag{5.66}$$

and

$$\begin{aligned} & 2f(S,T) \\ =\ & f(S \cap W_1, T \cap W_1) + f(S - W_1, T - W_1) \\ & + f(S \cap W_2, T \cap W_2) + f(S - W_2, T - W_2) \\ \geq\ & f(S \cap (W_1 \cap W_2), T \cap (W_1 \cap W_2)) + f(S \cap (W_1 \cup W_2), T \cap (W_1 \cup W_2)) \\ & + f(S - (W_1 \cup W_2), T - (W_1 \cup W_2)) + f(S - (W_1 \cap W_2), T - (W_1 \cap W_2)) \\ \geq\ & 2f(S,T). \end{aligned} \tag{5.67}$$

Hence, $W_1 \cup W_2, W_1 \cap W_2 \in \mathcal{W}$, i.e., $\mathcal{W}$ is closed with respect to union $\cup$ and intersection $\cap$. Furthermore, by the definition of a separator, $\mathcal{W}$ is complemented. This completes the proof of the present lemma. □

For a bisubmodular system $(\mathcal{F}, f)$ on V and a subset U of V such that there exists a signed subset $(X,Y) \in \mathcal{F}$ with $U = X \cup Y$, the *restriction* $(\mathcal{F}^U, f^U)$ of $(\mathcal{F}, f)$ to U is the bisubmodular system on U defined by

$$\mathcal{F}^U = \{(X,Y) \mid (X,Y) \in \mathcal{F}, X \cup Y \subseteq U\}, \tag{5.68}$$
$$f^U(X,Y) = f(X,Y) \quad ((X,Y) \in \mathcal{F}^U). \tag{5.69}$$

Theorem 5.23: *For a bisubmodular system $(\mathcal{F}, f)$ on V there uniquely exists a partition Π_0 of V such that the following three hold:*

(i) *For each $U \in \Pi_0$ and $(X,Y) \in \mathcal{F}$ we have $(X \cap U, Y \cap U) \in \mathcal{F}$.*

(ii) *Restrictions $(\mathcal{F}^U, f^U)$ $(U \in \Pi_0)$ are connected.*

(iii) *For any $(X,Y) \in \mathcal{F}$,*

$$f(X,Y) = \sum_{U \in \Pi_0} f^U(X \cap U, Y \cap U). \tag{5.70}$$

(Proof) The partition Π_0 is given by the set of the atoms of the Boolean lattice $\mathcal{W}$ formed by the separators of $(\mathcal{F}, f)$. The proof is almost the same as in the case of submodular systems (see [36, Section 3.3.d]). However, we include the proof for the sake of completeness.

Let $\Pi_0 = \{U_1, U_2, \cdots, U_k\}$ be the set of the atoms of the Boolean lattice $\mathcal{W}$ formed by the separators of $(\mathcal{F}, f)$. Let (X, Y) be any signed subset in $\mathcal{F}$. Then, we have

$$(X \cap U_i, Y \cap U_i) \in \mathcal{F} \quad (i = 1, \cdots, k). \tag{5.71}$$

Also, we have

$$\begin{aligned} & kf(X,Y) \\ = \ & f(X \cap U_1, Y \cap U_1) + \cdots + f(X \cap U_k, Y \cap U_k) \\ & +f(X - U_1, Y - U_1) + \cdots + f(X - U_k, Y - U_k) \\ \geq \ & f(X \cap U_1, Y \cap U_1) + \cdots + f(X \cap U_k, Y \cap U_k) + (k-1)f(X,Y), \end{aligned} \tag{5.72}$$

It follows that

$$f(X,Y) = f(X \cap U_1, Y \cap U_1) + \cdots + f(X \cap U_k, Y \cap U_k). \tag{5.73}$$

Let us consider the connectedness of any $(\mathcal{F}^{U_i}, f^{U_i})$ for $U_i \in \Pi_0$. Suppose that there exists a subset W of U_i such that for each $(X, Y) \in \mathcal{F}$ with $X \cup Y = U_i$ we have (5.57) and (5.58) with (S, T) being replaced by (X, Y). It follows from (5.73) that for any orthant $(S, T) \in \mathcal{F}$ we have

$$\begin{aligned} f(S,T) \ = \ & f(S \cap U_1, S \cap U_1) + \cdots + f(S \cap U_{i-1}, T \cap U_{i-1}) \\ & +f(S \cap W, T \cap W) + f(S \cap (U_i - W), T \cap (U_i - W)) \\ & +f(S \cap U_{i+1}, T \cap U_{i+1}) + \cdots + f(S \cap U_k, T \cap U_k) \\ \geq \ & f(S \cap W, T \cap W) + f(S - W, T - W) \\ \geq \ & f(S,T). \end{aligned} \tag{5.74}$$

Therefore, W must be a member of $\mathcal{W}$, and hence, $W = \emptyset$ or $W = U_i$ since U_i is an atom of $\mathcal{W}$.

Finally, we show that the uniqueness of the partition. Suppose that there exists another partition $\Pi' = \{U'_1, \cdots, U'_l\}$ such that (i)$\sim$(iii) in the present theorem hold. Then, by (iii) we have $\Pi' \subseteq \mathcal{W}$. Therefore, each $U'_i \in \Pi'$ can be represented as $U'_i = U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_m}$ for some $\{i_1, \cdots, i_m\} \subseteq \{1, \cdots, k\}$. It follows from (5.73) that we have

$$f(X,Y) = f(X \cap U_{i_1}, Y \cap U_{i_1}) + \cdots + f(X \cap U_{i_m}, Y \cap U_{i_m}) \tag{5.75}$$

for any $(X, Y) \in \mathcal{F}$ such that $X \cup Y \subseteq U_i'$. Since $(\mathcal{F}^{U_i'}, f^{U_i'})$ is connected, we must have $m = 1$. Hence, Π' is the set of all the atoms of $\mathcal{W}$ since Π' is a partition of V. $\square$

Each $(\mathcal{F}^U, f^U)$ for $U \in \Pi_0$ in Theorem 5.23 is called a *connected component* of $(\mathcal{F}, f)$.

The connected components of a bisubmodular system $(\mathcal{F}, f)$ on V are classified into two classes: fully connected ones and nonfully connected ones. Furthermore, Theorem 5.23 together with Theorem 5.20 shows that after an appropriate reflection any bisubmodular system is the direct sum of fully connected bisubmodular systems and connected submodular systems.

The following theorem characterizes the nonfully connected components of a bisubmodular system.

Theorem 5.24: *Suppose that* $(\mathcal{F}, f)$ *is a bisubmodular system on* V *and that* $\mathrm{P}_*(f) = \mathrm{F}(\mathcal{G})$ *for* $\mathcal{G} \in \mathbf{G}$*, where* $\mathbf{G}$ *is defined by* (5.47)*. The nonfully connected components of* $(\mathcal{F}, f)$ *are given by* $(\mathcal{F}^U, f^U)$ $(U \in \Pi(\mathcal{G}))$*.* $\square$

To show Theorem 5.24 we need some lemmas.

Lemma 5.25: *Suppose that* $\mathrm{P}_*(f) = \mathrm{F}(\mathcal{G})$ *for some* $\mathcal{G} \in \mathbf{G}$*. Then,* $\mathcal{G}$ *is closed with respect to reversal.*

(Proof) Let x be a vector in $\mathrm{P}_*(f)$ such that $\mathcal{F}(x) = \mathcal{G}$. For any $(X, Y) \in \mathcal{G}$ we have $x(X, Y) = f(X, Y)$. Since $y(X, Y) = f(X, Y)$ holds for any $y \in \mathrm{P}_*(f)$, it follows from the definition of signed saturation function that

$$X \cup Y \subseteq \mathrm{sat}^{(+)}(x) \cap \mathrm{sat}^{(-)}(x). \tag{5.76}$$

Therefore, similarly from (5.76) and (5.6)$\sim$(5.9) we have

$$\forall v \in Y\colon \quad \mathrm{dep}(x, +v) \sqsubseteq (Y, X), \tag{5.77}$$

$$\forall v \in X\colon \quad \mathrm{dep}(x, -v) \sqsubseteq (Y, X). \tag{5.78}$$

This implies $(Y, X) \in \mathcal{F}(x)$, i.e., $(Y, X) \in \mathcal{G}$. $\square$

From this lemma we can show

Lemma 5.26: *Suppose that* $\mathrm{P}_*(f) = \mathrm{F}(\mathcal{G})$ *for some* $\mathcal{G} \in \mathbf{G}$ *and that* $\mathrm{Supp}(\mathcal{G}) \neq \emptyset$*. Then,* $\mathcal{B}$ *defined by*

$$\mathcal{B} = \{X \cup Y \mid (X, Y) \in \mathcal{G}\} \tag{5.79}$$

is a Boolean lattice with respect to the set union and intersection as the lattice operations and has the set $\Pi(\mathcal{G})$ *of its atoms.*

(Proof) Since $\mathcal{G}$ is $\{\sqcup, \sqcap\}$-closed, for any $(X_i, Y_i) \in \mathcal{G}$ $(i = 1, 2)$ we have $((X_1, Y_1) \sqcup (X_2, Y_2)) \sqcup (X_1, Y_1) \in \mathcal{G}$. Denote this signed subset by (U_1, W_1). Then, $(X_1 \cup Y_1) \cup (X_2 \cup Y_2) = U_1 \cup W_1$. Hence, $\mathcal{B}$ is closed with respect to set union $\cup$. Also, since (Y_2, X_2) belongs to $\mathcal{G}$ due to Lemma 5.25, we have $((X_1, Y_1) \sqcap (X_2, Y_2)) \sqcup ((X_1, Y_1) \sqcap (Y_2, X_2)) \in \mathcal{G}$. Denote this signed subset (U_2, W_2). Then, $(X_1 \cup Y_1) \cap (X_2 \cup Y_2) = U_2 \cup W_2$. Hence, $\mathcal{B}$ is closed with respect to set intersection $\cap$. Moreover, for any $(X, Y) \in \mathcal{G}$ let (S, T) be a maximal element of $\mathcal{G}$ such that $(X, Y) \sqsubseteq (S, T)$. Note that $S \cup T = \mathrm{Supp}(\mathcal{G})$. Since $(Y, X) \in \mathcal{G}$ due to Lemma 5.25, we have $(S, T) \sqcup (Y, X) \in \mathcal{G}$. Denote this signed subset by (U_3, W_3). Then we have $(S \cup T) - (X \cup Y) = U_3 \cup W_3$, so that $\mathcal{B}$ is complemented with $\mathrm{Supp}(\mathcal{G})$ as the whole set, the maximum element. Therefore, $\mathcal{B}$ is a Boolean lattice with $\Pi(\mathcal{G})$ being the set of its atoms. □

(The proof of Theorem 5.24)
Suppose that $\mathcal{G} \in \mathbf{G}$ satisfies $\mathrm{F}(\mathcal{G}) = \mathrm{P}_*(f)$ and $\Pi(\mathcal{G}) = \{Q_1, \cdots, Q_l\}$. Let (S, T) be a maximal element of $\mathcal{G}$. From Lemma 5.26, for each $Q \in \Pi(\mathcal{G})$ we have $(S \cap Q, T \cap Q), (T \cap Q, S \cap Q) \in \mathcal{G}$. Then we have for any $(X, Y) \in \mathcal{F}$

$$\begin{aligned}
&(X \cap Q, Y \cap Q) \\
&\quad = ((S \cap Q, T \cap Q) \sqcap (X, Y)) \sqcup ((T \cap Q, S \cap Q) \sqcap (X, Y)) \in \mathcal{F}. \qquad (5.80)
\end{aligned}$$

Let R be the support of $\mathcal{G}$. Since $(S, T), (T, S) \in \mathcal{G}$, we have for any $(X, Y) \in \mathcal{F}$

$$(X - R, Y - R) = ((X, Y) \sqcup (S, T)) \sqcap ((X, Y) \sqcup (T, S)) \in \mathcal{F}. \qquad (5.81)$$

Moreover,

$$\begin{aligned}
f(S, T) &= f(S, T) + f(T \cap Q_1, S \cap Q_1) + \cdots + f(T \cap Q_l, S \cap Q_l) \\
&\quad + f(S \cap Q_1, T \cap Q_1) + \cdots + f(S \cap Q_l, T \cap Q_l) \\
&\geq f(S \cap Q_1, T \cap Q_1) + \cdots + f(S \cap Q_l, T \cap Q_l) \\
&\geq f(S, T). \qquad (5.82)
\end{aligned}$$

Hence, we have $f(S, T) = f(S \cap Q_1, T \cap Q_1) + \cdots + f(S \cap Q_l, T \cap Q_l)$. Similarly, we have $f(T, S) = f(T \cap Q_1, S \cap Q_1) + \cdots + f(T \cap Q_l, S \cap Q_l)$. Therefore,

$$\begin{aligned}
2f(X, Y) &= f(X, Y) + f(S, T) + f(X, Y) + f(T, S) \\
&= f(X, Y) + f(S \cap Q_1, T \cap Q_1) + \cdots + f(S \cap Q_l, T \cap Q_l) \\
&\quad + f(X, Y) + f(T \cap Q_1, S \cap Q_1) + \cdots + f(T \cap Q_l, S \cap Q_l) \\
&\geq f(S \cap X \cap Q_1, T \cap Y \cap Q_1) + \cdots + f(S \cap X \cap Q_l, T \cap Y \cap Q_l)
\end{aligned}$$

$$\begin{aligned}
&+f((S,T)\sqcup(X,Y))\\
&+f(T\cap X\cap Q_1, S\cap Y\cap Q_1)+\cdots+f(T\cap X\cap Q_l, S\cap Y\cap Q_l)\\
&+f((T,S)\sqcup(X,Y))\\
\geq\ & f(X\cap Q_1, Y\cap Q_1)+\cdots+f(X\cap Q_l, Y\cap Q_l)\\
&+f(X,Y)+f(X-R,Y-R)\\
\geq\ & f(X\cap R, Y\cap R)+f(X,Y)+f(X-R,Y-R)\\
\geq\ & 2f(X,Y). \qquad (5.83)
\end{aligned}$$

It follows from (5.83) that

$$f(X,Y)=f(X\cap Q_1, Y\cap Q_1)+\cdots+f(X\cap Q_l, Y\cap Q_l)+f(X-R,Y-R). \quad (5.84)$$

We see from (5.84) that each $Q\in\Pi(\mathcal{G})$ is a separator of $(\mathcal{F},f)$, i.e., a member of $\mathcal{W}$. For any $Q\in\Pi(\mathcal{G})$ there exist some atoms U_{i_j} $(j=1,\cdots,m)$ of $\mathcal{W}$ such that $Q=U_{i_1}\cup\cdots\cup U_{i_m}$. Since $Q\in\mathcal{B}$, there exists $(X,Y)\in\mathcal{F}$ with $X\cup Y=Q$ such that $(Y,X)\in\mathcal{F}$ and $f(X,Y)+f(Y,X)=0$. Then, it follows from Theorem 5.23 that

$$(X\cap U_{i_j}, Y\cap U_{i_j}),\ (Y\cap U_{i_j}, X\cap U_{i_j})\in\mathcal{F}\quad (j=1,\cdots,m) \qquad (5.85)$$

and

$$\begin{aligned}
0\ &=\ f(X,Y)+f(Y,X)\\
&=\ f(X\cap U_{i_1}, Y\cap U_{i_1})+\cdots+f(X\cap U_{i_m}, Y\cap U_{i_m})\\
&\quad\ +f(Y\cap U_{i_1}, X\cap U_{i_1})+\cdots+f(Y\cap U_{i_m}, X\cap U_{i_m})\\
&\geq\ 0. \qquad (5.86)
\end{aligned}$$

Therefore, we have

$$f(X\cap U_{i_j}, Y\cap U_{i_j})+f(Y\cap U_{i_j}, X\cap U_{i_j})=0\quad (j=1,\cdots,m). \qquad (5.87)$$

Hence, by the definition of $\Pi(\mathcal{G})$ we must have $m=1$, i.e., Q is an atom of $\mathcal{W}$. Also, it is clear that the connected component $(\mathcal{F}^Q, f^Q)$ is not fully connected.

Conversely, suppose that for some atom U of $\mathcal{W}$ the connected component $(\mathcal{F}^U, f^U)$ is not fully connected. Then, there exists $(X,Y)\in\mathcal{F}$ with $X\cup Y=U$ such that $(Y,X)\in\mathcal{F}$ and $f(X,Y)+f(Y,X)=0$. Then, we have $U\in\mathcal{B}$, and hence, $U=Q_{i_1}\cup\cdots\cup Q_{i_m}$ for some $Q_{i_j}\in\Pi(\mathcal{G})$ $(j=1,\cdots,m)$. It follows from (5.84) that for any $(Z,W)\in\mathcal{F}^U$ we have

$$f(Z,W)=f(Z\cap Q_{i_1}, W\cap Q_{i_1})+\cdots+f(Z\cap Q_{i_m}, W\cap Q_{i_m}). \qquad (5.88)$$

Since $(\mathcal{F}^U, f^U)$ is connected, we must have $m = 1$. Hence, we have $U \in \Pi(\mathcal{G})$.
(End of the proof of Theorem 5.24)

Unfortunately, given a vector $x \in \mathrm{P}_*(f)$, the exchangeability graph $G(\mathcal{F}(x))$ does not contain enough information about the partition Π_0 given in Theorem 5.23. However, we can enumerate the nonfully connected components of a bisubmodular system, i.e., we can find the partition $\Pi(\mathcal{G})$ given in Theorem 5.24. The procedure is outlined as follows.

We see from Theorem 5.20 and results in [15] and [36, Section 3.3.d] that $\Pi(\mathcal{G})$ is exactly the set of the vertex sets of the balanced components (see Section 2.3) of the exchangeability graph $G(\mathcal{F}(x))$ associated with x. We assume that $(\mathcal{F}, f)$ is simple without loss of generality. If we are given an extreme point x of $\mathrm{P}(f)$, then we can compute the bidirected graph $G(\mathcal{F}(x))$ (or, the signed poset $\mathcal{P}(x)$), using Algorithms I and II given in Section 5.3. Moreover, the balancedness of a bidirected graph can be checked efficiently by means of the breadth-first search. Hence, we can compute the nonfully connected components of a bisubmodular system in $\mathrm{O}(|V|^2)$ time, where we assume an oracle for the function evaluation of f.

Chapter 6

Characterizations of Bisubmodular Functions

As was seen in Theorem 5.12 bisubmodular functions are characterized by the validity of the greedy algorithm described in Section 5.4. In this chapter, we consider other characterizations of bisubmodular functions. It is well known that a set function $f : 2^V \to \mathbf{R}$ is submodular if and only if its derivatives are monotone nonincreasing (see, e.g., [54]). Our results are generalizations of this fact.

This chapter is mainly based on the collaborated work [7] with S. Fujishige and T. Naitoh.

6.1. A Characterization of Bimodular Functions

Suppose that $\mathcal{F}$ is a simple and spanning $\{\sqcup, \sqcap\}$-closed family on V. Recall that a function $f : \mathcal{F} \to \mathbf{R}$ on $\mathcal{F}$ is called bimodular if (2.13) holds with equality for each $(X_i, Y_i) \in \mathcal{F}$ $(i = 1, 2)$. Bimodular functions are characterized as follows.

Lemma 6.1: *Suppose that f is a bimodular function on a simple and spanning $\{\sqcup, \sqcap\}$-closed family $\mathcal{F}$ on V with $f(\emptyset, \emptyset) = 0$. Then there exists a unique vector $\nu \in \mathbf{R}^V$ such that*

$$f(X, Y) = \sum_{v \in X} \nu(v) - \sum_{v \in Y} \nu(v). \tag{6.1}$$

(Proof) Let $\mathcal{C} : (\emptyset, \emptyset) = (S \cap U_0, T \cap U_0) \sqsubset (S \cap U_1, T \cap U_1) \sqsubset \cdots \sqsubset (S \cap U_n, T \cap U_n) = (S, T)$ be any maximal chain of $\mathcal{F}$, where note that $\emptyset = U_0 \subset U_1 \subset \cdots \subset U_n = V$ and $|U_i - U_{i-1}| = 1$ $(i = 1, 2, \cdots, n)$ with $n = |V|$ due to the assumption. For $i = 1, \cdots, n$ let $v_i = U_i - U_{i-1}$. Define a vector $\nu \in \mathbf{R}^V$ by

$$\nu(v_i) = \begin{cases} f(S \cap U_i, T \cap U_i) - f(S \cap U_{i-1}, T \cap U_{i-1}) & (\text{if } v_i \in S) \\ f(S \cap U_{i-1}, T \cap U_{i-1}) - f(S \cap U_i, T \cap U_i) & (\text{if } v_i \in T) \end{cases} \tag{6.2}$$

for $i = 1, \cdots, n$. We can easily show that for any $(X, Y) \in \mathcal{F}$ with $(X, Y) \sqsubseteq (S, T)$ equation (6.1) holds (see [36, Lemma 7.5]). Then we also have for any $(X, Y) \in \mathcal{F}$

$$f(X, Y) \;\; = \;\; -f(S, T) + f(S - Y, T - X) + f(S \cap X, T \cap Y)$$

$$\begin{aligned}
&= -\sum_{v\in S}\nu(v)+\sum_{v\in T}\nu(v)+\sum_{v\in S-Y}\nu(v)-\sum_{v\in T-X}\nu(v)\\
&\quad +\sum_{v\in S\cap X}\nu(v)-\sum_{v\in T\cap Y}\nu(v)\\
&= \sum_{v\in X}\nu(v)-\sum_{v\in Y}\nu(v), \qquad (6.3)
\end{aligned}$$

where the first equality follows from the bimodularity of f on $\mathcal{F}$ and note that (6.1) holds for $(X,Y)=(S,T),(S-Y,T-X),(S\cap X,T\cap Y)$ with $(X,Y)\sqsubseteq(S,T)$. Moreover, since $\mathcal{F}$ is simple and spanning, such a representation of f is unique. □

We see from Lemma 6.1 that the problem of minimizing bimodular functions is equivalent to the minimum-weight ideal problem for signed posets.

6.2. A Characterization of Bisubmodular Functions

Suppose that $\mathcal{F}$ is a simple and spanning $\{\sqcup,\sqcap\}$-closed family on V. A function $f:\mathcal{F}\to\mathbf{R}$ is called *bisubmodular in an orthant* (S,T) if for each $X_1,X_2\subseteq S$ and $Y_1,Y_2\subseteq T$ we have (2.13). It should be noted that $f:\mathcal{F}\to\mathbf{R}$ is bisubmodular in an orthant (S,T) if and only if a set function $f':\mathcal{D}_{(S,T)}\to\mathbf{R}$ defined by

$$f'(X)=f(X\cap S,X\cap T)\qquad (X\in\mathcal{D}_{(S,T)}) \tag{6.4}$$

is an ordinary submodular function on $\mathcal{D}_{(S,T)}$, where $\mathcal{D}_{(S,T)}\subseteq 2^V$ is the distributed lattice defined by

$$\mathcal{D}_{(S,T)}=\{X\cup Y\mid (X,Y)\in\mathcal{F},(X,Y)\sqsubseteq(S,T)\}. \tag{6.5}$$

We say two signed subsets (X_1,Y_1) and (X_2,Y_2) of V are *adjacent* if

$$(X_1,Y_1)=(X\cup\{v\},Y),\ (X_2,Y_2)=(X,Y\cup\{v\}) \tag{6.6}$$

for some $(X,Y)\in 3^V$ and $v\in V-(X\cup Y)$.

For any function $f:\mathcal{F}\to\mathbf{R}$ on a simple and spanning $\{\sqcup,\sqcap\}$-closed family $\mathcal{F}$ on V, let us consider the following two conditions.

(i) f is bisubmodular in each orthant.
(ii) For any two adjacent orthants $(S_1,T_1),(S_2,T_2)\in\mathcal{F}$ we have

$$\begin{aligned}
f(S_1,T_1)+f(S_2,T_2) &\geq 2f((S_1,T_1)\sqcup(S_2,T_2)) &(6.7)\\
&(= 2f((S_1,T_1)\sqcap(S_2,T_2))). &(6.8)
\end{aligned}$$

Lemma 6.2: *Conditions* (i) *and* (ii) *are equivalent to Condition* (i) *and the following condition:*

(ii') *For any two adjacent signed subsets* $(X_1, Y_1), (X_2, Y_2) \in \mathcal{F}$ *we have*

$$
\begin{aligned}
f(X_1, Y_1) + f(X_2, Y_2) &\geq 2f((X_1, Y_1) \sqcup (X_2, Y_2)) && (6.9)\\
(&= 2f((X_1, Y_1) \sqcap (X_2, Y_2))). && (6.10)
\end{aligned}
$$

(Proof) It suffices to prove that Conditions (i) and (ii) imply Condition (ii').

Suppose that $(X_1, Y_1) = (X \cup \{v\}, Y)$ and $(X_2, Y_2) = (X, Y \cup \{v\})$ for some $(X, Y) \in \mathcal{F}$ and $v \notin X \cup Y$. Let us consider the signed poset $\mathcal{P}(\mathcal{F}) = (V, A; \partial)$. Since $(X \cup \{v\}, Y) \in \mathcal{F}$, there exists an orthant $(Z \cup \{v\}, W) \in \mathcal{F}$ such that $(X \cup \{v\}, Y) \sqsubseteq (Z \cup \{v\}, W)$. Then, $(Z, W \cup \{v\})$ is also an orthant in $\mathcal{F}$ since

$$(Z, W \cup \{v\}) = ((Z \cup \{v\}, W) \sqcup (X, Y \cup \{v\})) \sqcup (X, Y \cup \{v\}). \tag{6.11}$$

We have from Condition (i) that

$$f(X \cup \{v\}, Y) + f(Z, W) \geq f(Z \cup \{v\}, W) + f(X, Y) \tag{6.12}$$

and

$$f(X, Y \cup \{v\}) + f(Z, W) \geq f(Z, W \cup \{v\}) + f(X, Y). \tag{6.13}$$

Hence, we have

$$
\begin{aligned}
& f(X \cup \{v\}, Y) + f(X, Y \cup \{v\}) - 2f(X, Y)\\
&\quad \geq f(Z \cup \{v\}, W) + f(Z, W \cup \{v\}) - 2f(Z, W)\\
&\quad \geq 0, && (6.14)
\end{aligned}
$$

where the last inequality is due to (ii). This completes the proof of the present lemma.□

Lemma 6.3: *Suppose that* $\mathcal{F}$ *is a simple and spanning* $\{\sqcup, \sqcap\}$*-closed family on* V. *Then, for any pairwise disjoint* $X, Y, A, B \subseteq V$ *such that* $(X \cup A, Y \cup B), (X \cup B, Y \cup A) \in \mathcal{F}$ *and* $A \cup B \neq \emptyset$ *we have*

$$(X \cup (B \cup \{v\}), Y \cup (A - \{v\})) \in \mathcal{F} \textit{ for some } v \in A \tag{6.15}$$

or

$$(X \cup (B - \{v\}), Y \cup (A \cup \{v\})) \in \mathcal{F} \textit{ for some } v \in B. \tag{6.16}$$

(Proof) Since $(X, Y), (X \cup A, Y \cup B) \in \mathcal{F}$ and $\mathcal{F}$ is simple, we have

$$(X \cup \{v\}, Y) \in \mathcal{F} \text{ for some } v \in A \tag{6.17}$$

or

$$(X, Y \cup \{v\}) \in \mathcal{F} \text{ for some } v \in B. \tag{6.18}$$

In the former case, we have

$$(X \cup (B \cup \{v\}), Y \cup (A - \{v\})) = ((X \cup B, Y \cup A) \sqcup (X \cup \{v\}, Y)) \sqcup (X \cup \{v\}, Y) \in \mathcal{F}. \tag{6.19}$$

In the latter case, we have

$$(X \cup (B - \{v\}), Y \cup (A \cup \{v\})) = ((X \cup B, Y \cup A) \sqcup (X, Y \cup \{v\})) \sqcup (X, Y \cup \{v\}) \in \mathcal{F}. \tag{6.20}$$

□

Lemma 6.4: *Conditions* (i) *and* (ii') *are equivalent to Condition* (i) *and the following condition:*

(ii") *For any pairwise disjoint* $X, Y, A, B \subseteq V$ *such that* $(X \cup A, Y \cup B), (X \cup B, Y \cup A) \in \mathcal{F}$ *we have*

$$f(X \cup A, Y \cup B) + f(X \cup B, Y \cup A) \geq 2f(X, Y). \tag{6.21}$$

(Proof) It suffices to prove that Conditions (i) and (ii') imply Condition (ii"). We prove this lemma by induction on $k = |A \cup B|$. In the case when $k = 0$, we have $A = B = \emptyset$ and (6.21) holds trivially. When $k = 1$, we have (6.21) from (ii'). Suppose that $k > 1$ and inequality (6.21) holds for any pairwise disjoint $X, Y, A, B \subseteq V$ such that $(X \cup A, Y \cup B), (X \cup B, Y \cup A) \in \mathcal{F}$ with $|A \cup B| = k - 1$.

It follows from Lemma 6.3 that either (6.15) or (6.16) holds. Suppose that (6.15) holds. Let $A' = A - \{v\}$ and $B' = B$. We have

$$((X \cup \{v\}) \cup A', Y \cup B') = (X \cup A, Y \cup B) \in \mathcal{F} \tag{6.22}$$

and

$$((X \cup \{v\}) \cup B', Y \cup A') = (X \cup (B \cup \{v\}), Y \cup (A - \{v\})) \in \mathcal{F}. \tag{6.23}$$

Since $|A' \cup B'| = k - 1$, we have by the induction hypothesis that

$$f(X \cup A, Y \cup B) + f(X \cup (B \cup \{v\}), Y \cup (A - \{v\})) \geq 2f(X \cup \{v\}, Y). \tag{6.24}$$

Also, from (ii') we have

$$f(X \cup B, Y \cup A) + f(X \cup (B \cup \{v\}), Y \cup (A - \{v\})) \geq 2f(X \cup B, Y \cup (A - \{v\})). \quad (6.25)$$

On the other hand, we have from (i) that

$$f(X \cup \{v\}, Y) + f(X \cup B, Y \cup (A - \{v\})) \geq f(X \cup (B \cup \{v\}), Y \cup (A - \{v\})) + f(X, Y). \quad (6.26)$$

Hence, it follows from (6.24)∼(6.26) that

$$\begin{aligned} & f(X \cup A, Y \cup B) + f(X \cup B, Y \cup A) + 2f(X \cup (B \cup \{v\}), Y \cup (A - \{v\})) \\ \geq\ & 2f(X \cup \{v\}, Y) + 2f(X \cup B, Y \cup (A - \{v\})) \\ \geq\ & 2f(X \cup (B \cup \{v\}), Y \cup (A - \{v\})) + 2f(X, Y). \end{aligned} \quad (6.27)$$

Therefore, the inequality (6.21) holds when $|A \cup B| = k$. The case when (6.16) holds can be treated similarly. □

Theorem 6.5: *Suppose that $\mathcal{F}$ is a simple and spanning $\{\sqcup, \sqcap\}$-closed family on V. A function $f : \mathcal{F} \to \mathbf{R}$ is bisubmodular if and only if Conditions* (i) *and* (ii) *hold.*
(Proof) The "only if" part is trivial. We prove the "if" part. Let $(X_i, Y_i) \in \mathcal{F}$ $(i = 1, 2)$. Putting $A = X_2 \cap Y_1$, $B = X_1 \cap Y_2$ and $(X, Y) = (X_1, Y_1) \sqcup (X_2, Y_2)$, we have

$$(X \cup A, Y \cup B) = ((X_1 \cup X_2) - Y_2, (Y_1 \cup Y_2) - X_2) \in \mathcal{F} \quad (6.28)$$

and

$$(X \cap B, Y \cup A) = ((X_1 \cup X_2) - Y_1, (Y_1 \cup Y_2) - X_1) \in \mathcal{F}. \quad (6.29)$$

It follows from Lemma 6.4 that

$$\begin{aligned} & f((X_1 \cup X_2) - Y_2, (Y_1 \cup Y_2) - X_2) + f((X_1 \cup X_2) - Y_1, (Y_1 \cup Y_2) - X_1) \\ \geq\ & 2f((X_1, Y_1) \sqcup (X_2, Y_2)). \end{aligned} \quad (6.30)$$

On the other hand, we have from Condition (i) that and

$$f(X_1, Y_1) + f((X_1, Y_1) \sqcup (X_2, Y_2)) \geq f((X_1 \cup X_2) - Y_1, (Y_1 \cup Y_2) - X_1) + f(X_1 - Y_2, Y_1 - X_2) \quad (6.31)$$

and

$$f(X_2, Y_2) + f((X_1, Y_1) \sqcup (X_2, Y_2)) \geq f((X_1 \cup X_2) - Y_2, (Y_1 \cup Y_2) - X_2) + f(X_2 - Y_1, Y_2 - X_1). \quad (6.32)$$

Consequently, from inequalities (6.30)∼(6.32) we obtain

$$\begin{aligned} & f(X_1,Y_1)+f(X_2,Y_2)+2f((X_1,Y_1)\sqcup(X_2,Y_2)) \\ \geq\ & f((X_1\cup X_2)-Y_1,(Y_1\cup Y_2)-X_1)+f((X_1\cup X_2)-Y_2,(Y_1\cup Y_2)-X_2) \\ & +f(X_1-Y_2,Y_1-X_2)+f(X_2-Y_1,Y_2-X_1) \\ \geq\ & 3f((X_1,Y_1)\sqcup(X_2,Y_2))+f((X_1,Y_1)\sqcap(X_2,Y_2)) \end{aligned} \tag{6.33}$$

from which follows the desired inequality. □

As we had seen in Theorem 5.12, the bisubmodularity is also characterized by the validity of the greedy algorithm given in Section 5.4. Hence, we can show Theorem 6.5 via the validity of the greedy algorithm (see [38]).

Define

$$\mathbf{D}=\{(x,y)\mid x,y\in\mathbf{Z}_+,x+y\leq|V|\}, \tag{6.34}$$

where $\mathbf{Z}_+$ denotes the set of nonnegative integers. A function $f:3^V\to\mathbf{R}$ is said to be *cardinality-symmetric* if its function value $f(X,Y)$ depends only on the cardinalities $|X|$ and $|Y|$ of its arguments, i.e., there exists a function $g\colon\mathbf{D}\to\mathbf{R}$ such that

$$f(X,Y)=g(|X|,|Y|) \tag{6.35}$$

for all $(X,Y)\in 3^V$.

Define operators Δ_1 and Δ_2 as follows. For any function g on $\mathbf{D}$,

$$\Delta_1 g(x,y) = g(x+1,y)-g(x,y), \tag{6.36}$$

$$\Delta_2 g(x,y) = g(x,y+1)-g(x,y) \tag{6.37}$$

for $x,y\in\mathbf{Z}_+$ with $x+y\leq|V|-1$.

For any bivariate function h we say that h is *monotone nonincreasing* if $h(x,y)\geq h(z,w)$ for every $x\leq z$ and $y\leq w$ in the domain.

From Theorem 6.5 we have

Corollary 6.6: *A cardinality-symmetric function $f:3^V\to\mathbf{R}$ is bisubmodular if and only if f is expressed as* (6.35) *in terms of a bivariate function $g\colon\mathbf{D}\to\mathbf{R}$ that satisfies the following two conditions:*

(i) *For each $i=1,2$ $\Delta_i g$ is monotone nonincreasing.*

(ii) *For any $x,y\in\mathbf{Z}_+$ with $x+y=|V|-1$,*

$$g(x+1,y)+g(x,y+1)\geq 2g(x,y). \tag{6.38}$$

□

Chapter 7

Fractional Degree-two Polytopes and Ideal Polytopes

The concept of degree-two inequalities is introduced by E. L. Johnson and M. W. Padberg [50]. They noticed that there exists a natural correspondence between bidirected graphs and degree-two inequalities.

An inequality in n variables $x_1, \cdots, x_n$ is of *degree-two* if it is either $x_i + x_j \leq 1$, $-x_i - x_j \leq -1$ or $x_i - x_j \leq 0$ for some $i, j = 1, \cdots, n$. For example, the following is a system of degree-two inequalities:

$$\begin{array}{rrrrcr} -2x_1 & & & & \leq & -1, \\ -x_1 & +x_2 & & & \leq & 0, \\ & x_2 & +x_3 & & \leq & 1, \\ & x_2 & -x_3 & & \leq & 0, \\ & & x_3 & +x_4 & \leq & 1, \\ & & x_3 & -x_4 & \leq & 0, \\ & & 2x_3 & & \leq & 1. \end{array} \tag{7.1}$$

The 0-1 solutions of degree-two inequalities are of special interest. The stable sets, the node covers, the order ideals of a (directed) graph are described as the 0-1 solutions of systems of degree-two inequalities. Other examples of the 0-1 solutions of a degree-two inequalities can be found in many contexts, such as quadratic Boolean equations [42], linear and quadratic 0-1 programming [41], and logic [75]. In studies of quadratic Boolean equation (see e.g., [12] and [42]), bidirected graphs appeared implicitly as *implication graphs*, which was called signed covering graphs by T. Zaslavsky [86]. It should be also noted that a system of degree-two constraints is, in disguise, a *complete set of implicants* with their lengths at most two (see [45], [44], [46]).

In [50] Johnson and Padberg considered the polytope defined as the convex hull of the 0-1 solutions of a system of degree-two inequalities as a generalization of a stable set polytope and gave a class of facet-inducing inequalities. In this chapter, we consider a relaxation of the 0-1 solutions of degree-two inequalities, namely, we consider the solution set of degree-two inequalities and the inequalities $0 \leq x_j \leq 1$ ($j = 1, \cdots, n$), which we call a fractional degree-two polytope.

In Section 7.1, we show that there is an isomorphism between a fractional degree-two polytope and an ideal polytope. Also, the integrality of the ideal polytope of any bidirected graph is shown, and the half-integrality of any fractional degree-two polytope follows. It will be shown that the linear programs over fractional degree-two polytopes can be reduced to minimum-weight ideal problems for bidirected graphs. In Section 7.2, we give some necessary and sufficient condition for a fractional degree-two polytope to be integral. Finally in Section 7.3, we consider some set systems characterized by complete sets of implicants as an application.

This chapter is based on the paper [3].

7.1. Fractional Degree-two Polytopes and Ideal Polytopes

An *ideal polytope* $\mathrm{IP}(G)$ associated with a bidirected graph $G = (V, A; \partial)$ is defined as the solution set of the system

$$\langle \partial a, x\rangle \leq 0 \quad (a \in A), \tag{7.2}$$

$$-1 \leq x(v) \leq 1 \quad (v \in V) \tag{7.3}$$

of inequalities. Note that the integral points of $\mathrm{IP}(G)$ are precisely the characteristic vectors of the ideals of G.

For a system $\sum_{l=1}^{n} \gamma_{kl} x_l \leq \beta_k$ $(k = 1, \cdots, m)$ of degree-two inequalities in n variables $x_1, \cdots, x_n$ we naturally associate a bidirected graph $G = (V, A; \partial)$ as follows ([50]): $V = \{v_1, \cdots, v_n\}$, $A = \{a_1, \cdots, a_m\}$ and ∂ is defined as

$$\partial a_k = \begin{cases} v_i + v_j \\ -v_i - v_j \\ v_i - v_j \end{cases} \text{if} \begin{cases} \text{the } k\text{-th inequality is } x_i + x_j \leq 1 \\ \text{the } k\text{-th inequality is } -x_i - x_j \leq -1 \\ \text{the } k\text{-th inequality is } x_i - x_j \leq 0 \end{cases} \quad (k = 1, \cdots, m). \tag{7.4}$$

For our example (7.1) the bidirected graph in Figure 1.2 corresponds. Now the system $\sum_{l=1}^{n} \gamma_{kl} x_l \leq \beta_k$ $(k = 1, \cdots, m)$ is described in terms of bidirected graph G as

$$\langle \partial a, x\rangle \ \leq \ \frac{1}{2}\langle \partial a, \mathbf{1}_V\rangle \quad (a \in A), \tag{7.5}$$

where $\mathbf{1}_V \in \mathbf{R}^V$ is defined by $\mathbf{1}_V(v) = 1$ for $v \in V$, $\langle \cdot, \cdot \rangle$ is the (canonical) inner product, and ∂a should be regarded as a vector in $\mathbf{R}^V$. Conversely, given any bidirected graph $G = (V, A; \partial)$, the system (7.5) of inequalities is of degree-two. Hence, from now on, we always associate a bidirected graph G with a system of degree-two inequalities.

Given a bidirected graph $G = (V, A; \partial)$, we call the solution set of the system

$$\langle \partial a, x \rangle \ \leq \ \tfrac{1}{2}\langle \partial a, \mathbf{1}_V \rangle \quad (a \in A), \tag{7.6}$$

$$0 \leq x(v) \leq 1 \quad (v \in V) \tag{7.7}$$

of inequalities the *fractional degree-two polytope* associated with G and denote it by FD2P(G). It should be noted that if all the arcs of G are of type (1) then G can be regarded as an undirected graph and FD2P(G) is the *fractional stable set polytope* of G. Also, if all the arcs of G are of type (3), then G is an ordinary directed graph and FD2P(G) is the *order ideal polytope* of G.

It follows from the definitions that

Proposition 7.1: *For any bidirected graph $G = (V, A; \partial)$ we have $x \in \mathrm{FD2P}(G)$ if and only if $2x - \mathbf{1}_V \in \mathrm{IP}(G)$.* □

Also, we can easily see the following.

Lemma 7.2: *For any bidirected graph $G = (V, A; \partial)$ and $w : V \to \mathbf{R}$ x is an optimal solution for $\min\{\sum_{v \in V} w(v)x(v) \mid x \in \mathrm{FD2P}(G)\}$ if and only if $2x - \mathbf{1}_V$ is an optimal solution for $\min\{\sum_{v \in V} w(v)x(v) \mid x \in \mathrm{IP}(G)\}$.* □

Note that the mapping $x \mapsto 2x - \mathbf{1}_V$ gives an isomorphism between two polytopes FD2P(G) and IP(G), that the half-integral points in FD2P(G) correspond to the ideals of G and that the integral points in FD2P(G) correspond to the ideals (S, T) of G such that $S \cup T = V$.

The following theorem was implicitly given in [73, Proposition 3.1] and also follows from Theorem 5.10.

Theorem 7.3: *For any bidirected graph $G = (V, A; \partial)$ the ideal polytope* IP(G) *is integral.*

(Proof) Given a bidirected graph $G = (V, A; \partial)$ suppose that x is in IP(G). It suffices to show x is a convex combination of the characteristic vectors $\chi_{(X,Y)}$ $((X, Y) \in \mathcal{I}(G))$.

Let the distinct positive values of $|x(v)|$ $(v \in V)$ be given by

$$x_1 > x_2 > \cdots > x_p \ (> 0). \tag{7.8}$$

Define

$$\left.\begin{array}{rcl} U_i & = & \{v \mid v \in V, x(v) \geq x_i\} \\ W_i & = & \{v \mid v \in V, -x(v) \geq x_i\} \end{array}\right\} \quad (i = 1, 2, \cdots, p). \tag{7.9}$$

Since $x \in \mathrm{IP}(G)$, for each $i = 1, 2, \cdots, p$ we have $(U_i, W_i) \in \mathcal{I}(G)$. Define λ_i $(i = 0, 1, \cdots, p)$ by

$$\lambda_i = \begin{cases} 1 - x_1 & (i = 0) \\ x_i - x_{i+1} & (i = 1, \cdots, p-1) \\ x_p & (i = p) \end{cases} \tag{7.10}$$

Putting $(U_0, W_0) = (\emptyset, \emptyset)$ we have $x = \sum_{i=0}^{p} \lambda_i \chi_{(U_i, W_i)}$. Since $\sum_{i=0}^{p} \lambda_i = 1$ and $\lambda_i \geq 0$ $(i = 0, \cdots, p)$, the present theorem follows. □

The following is a generalization of a result of Nemhauser and Trotter [64] for the fractional stable set polytopes.

Corollary 7.4: *For any bidirected graph $G = (V, A; \partial)$ the fractional degree-two polytope* FD2P(G) *is half-integral.*

(Proof) The present corollary follows from Proposition 7.1 and Theorem 7.3. □

Corollary 7.5: *For any bidirected graph G the linear programming problem over the polytope* FD2P(G) *can be reduced to the minimum-weight ideal problem for G, and vice versa.*

(Proof) The present corollary follows from Lemma 7.2 and Theorem 7.3 □

It follows from the remark given after Lemma 3.19 that a linear programming problem over a fractional degree-two polytope can be reduced to a minimum-cut problem.

7.2. Characterizations for a Fractional Degree-two Polytope to be Integral

In this section we argue conditions for a fractional degree-two polytopes to be integral. It is well-know that a necessary and sufficient condition for a fractional stable set polytope to be integral is that the given undirected graph is bipartite. Also, if the bidirected graph G is an ordinary directed graph (i.e., all the arcs of G are of type (3)), the FD2P(G) is the order ideal polytope and known to be integral. These facts suggest the following generalization and is indeed true.

Theorem 7.6: *The fractional degree-two polytope* FD2P(G) *associated with a bidirected graph $G = (V, A; \partial)$ is integral if and only if G is balanced.*

(Proof) The sufficiency easily follows from Lemma 2.4 since the left-hand side matrix of defining inequalities (7.6) is totally unimodular if (and only if) G is balanced.

To see the necessity, suppose that for a bidirected graph $G = (V, A; \partial)$ the associated FD2P(G) is integral. Then, it follows from Lemma 7.2 together with the remarks given after it that for any weight function there exists a minimum-weight ideal (S,T) such that $S \cup T = V$. Define a weight function $w\colon V \to \mathbf{R}$ on V by

$$w = -\sum_{a \in A} \partial a, \tag{7.11}$$

where we regard ∂a as a vector in $\mathbf{R}^V$. Then, for any ideal (X,Y) of G we have

$$\begin{aligned} w(X) - w(Y) &= \langle -\sum_{a \in A} \partial a, (X,Y)\rangle \\ &= -\sum_{a \in A} \langle \partial a, (X,Y)\rangle \\ &\geq 0. \end{aligned} \tag{7.12}$$

Let (S,T) be an optimal ideal for Problem $\min\{w(X) - w(Y) \mid (X,Y) \in \mathcal{I}(G)\}$ such that $S \cup T = V$. Since $(\emptyset, \emptyset) \in \mathcal{I}(G)$ and $w(\emptyset) - w(\emptyset) = 0$, the optimal value is 0 and we have $w(S) - w(T) = 0$. Then, we have $\langle \partial a, (S,T)\rangle = 0$ for any $a \in A$. Hence, $\langle \partial a, (T,S)\rangle = 0$ for any $a \in A$ and the signed subset (T,S) is also an (optimal) ideal of G. It follows from Lemma 2.8 that G is balanced. □

The following corollary is easily derived from Hoffman and Kruskal's theorem.

Corollary 7.7: *A fractional degree-two polytope* FD2P(G) *is integral if and only if for any integral $l, u \in \mathbf{Z}^V$ and $c \in \mathbf{Z}^A$ the polytope described by the system*

$$\langle \partial a, x\rangle \leq c(a) \quad (a \in A), \tag{7.13}$$

$$l(v) \leq x(v) \leq u(v) \quad (v \in V) \tag{7.14}$$

of inequalities is integral. □

As a consequence of Theorem 7.3 and Lemma 2.7 we have the following.

Lemma 7.8: *Suppose that $G = (V, A; \partial)$ is a bidirected graph and U is a subset of V. We have* IP(G):U = IP$(G$:$U)$. □

For any $Q \subseteq [0,1]^V$ and $U \subseteq V$ define the *negation* $Q \,!\, U \subseteq [0,1]^V$ of Q at U by

$$Q \,!\, U = \{x \,!\, U \mid x \in Q\}, \tag{7.15}$$

where $x \,!\, U \in [0,1]^V$ is defined by

$$(x \,!\, U)(v) = \begin{cases} 1 - x(v) & (\text{if } v \in U) \\ x(v) & (\text{otherwise}) \end{cases} \qquad (v \in V). \tag{7.16}$$

Corollary 7.9: *The fractional degree-two polytope* FD2P(G) *associated with a bidirected graph* $G = (V, A; \partial)$ *is integral if and only if* FD2P(G) ! U *is an order ideal polytope for some* $U \subseteq V$.

(Proof) Sufficiency is trivial. Suppose that the fractional degree-two polytope FD2P(G) is integral. Then, by Theorem 7.6, $G = (V, A; \partial)$ is balanced, i.e., for some $U \subseteq V$ the reflection $G:U$ of G is an ordinary directed graph and we have $\mathrm{IP}(G):U = \mathrm{IP}(G:U)$ by Lemma 7.8. It follows from Proposition 7.1 that FD2P$(G)\,!\,U$ = FD2P$(G:U)$. Since FD2P$(G:U)$ is an order ideal polytope, the corollary holds. □

7.3. An Application

In this section we consider set systems characterized by "complete sets of implicants", which was developed by Hausmann and Korte [45] and Hausmann [44].

We call a pair $(V, \mathcal{F})$ of a nonempty finite set V and $\mathcal{F} \subseteq 2^V$ a *discrete system*. An *implicant* of discrete system $(V, \mathcal{F})$ is $(J^+, J^-) \in 3^V$ such that for any $F \subseteq V$ we have

$$J^+ \subseteq F, J^- \cap F = \emptyset \Longrightarrow F \notin \mathcal{F}. \tag{7.17}$$

The *length* of implicant (J^+, J^-) is $|J^+ \cup J^-|$. A set of implicants $\mathcal{J}$ with the property

$$F \notin \mathcal{F} \Longleftrightarrow \exists (J^+, J^-) \in \mathcal{J} : J^+ \subseteq F, J^- \cap F = \emptyset \tag{7.18}$$

is called a *complete set of implicants* of $(V, \mathcal{F})$.

If $\mathcal{J}$ is the complete set of implicants of a discrete system $(V, \mathcal{F})$ and all the implicants in $\mathcal{J}$ have lengths two, then we can associate a bidirected graph $G = (V, A; \partial)$ in an obvious way, i.e., A and ∂ are defined by

$$A = \{a_J \mid J = (J^+, J^-) \in \mathcal{J}\} \tag{7.19}$$

and

$$\partial a_J = \chi_J \quad (J = (J^+, J^-) \in \mathcal{J}) \tag{7.20}$$

and we see that $\mathcal{F}$ is the integral solutions of FD2P(G).

For $F_1, F_2 \in \mathcal{F}$ the *reduced system* with respect to F_1 and F_2 is the discrete system $(F_1 \Delta F_2, \tilde{\mathcal{F}})$, where Δ denotes the symmetric difference and

$$\tilde{\mathcal{F}} = \{F \mid F \subseteq F_1 \Delta F_2, F \cup (F_1 \cap F_2) \in \mathcal{F}\}. \tag{7.21}$$

For a discrete system $(V, \mathcal{F})$ define P$(\mathcal{F})$ as the convex hull of the characteristic vectors χ_F $(F \in \mathcal{F})$.

Theorem 7.10 (Ikebe and Tamura [46]): *Suppose that* $(V, \mathcal{F})$ *is a discrete system and* $F_1, F_2 \in \mathcal{F}$*. If there exists a complete set of implicants* $\tilde{\mathcal{J}}$ *of the reduced system* $(F_1 \Delta F_2, \tilde{\mathcal{F}})$ *such that all the lengths of implicants in* $\tilde{\mathcal{J}}$ *are two, then* $\mathrm{P}(\tilde{\mathcal{F}})$ *is a negation of an order ideal polytope.*

(Proof) If $\tilde{\mathcal{J}}$ is the complete set of implicants of $(F_1 \Delta F_2, \tilde{\mathcal{F}})$ and all the implicants in $\tilde{\mathcal{J}}$ have lengths two, then we can associate a bidirected graph $G = (F_1 \Delta F_2, A; \partial)$ by (7.19) and (7.20), where $\mathcal{J}$ being replaced with $\tilde{\mathcal{J}}$. Then, $\tilde{\mathcal{F}}$ is the integral solutions of FD2P(G). Furthermore, since $F_1 - F_2, F_2 - F_1 \in \tilde{\mathcal{F}}$, the bidirected graph G must be balanced. It follows from Theorem 7.6 that FD2P(G) is integral, and hence, FD2P$(G) = \mathrm{P}(\tilde{\mathcal{F}})$. We completes the proof of the present theorem using Corollary 7.9. $\square$

Chapter 8

Concluding Remarks

In this final chapter we review the results given in each main chapter and discuss some related topics.

In Chapter 3 we considered classical combinatorial optimization problems on bidirected graphs and networks such as the problems of finding the transitive closure and the Hasse diagram, the minimum cost circulation problem, the maximum flow problem, the minimum cut problem and the minimum-weight ideal problem. All of the problems can be reduced to those problems on ordinary directed graphs, namely, on their signed covering graphs. Here, the maxflow-mincut theorem was established for bidirected networks. Also, the half-integrality of optimal circulations was shown. Investigations of a condition for a bidirected network in which there always exists an integral optimal circulation may be an interesting research topic.

We had observed that the bisubmodular polyhedra associated with the cut functions of bidirected networks are actually projections of base polyhedra associated with the cut functions of their signed covering networks. That is, the bisubmodular polyhedra associated with the cut functions can be represented by base polyhedra. A naturally raised question is whether every bisubmodular polyhedron can be represented by base polyhedra or submodular polyhedra or not.

A novel algorithm for finding a minimum nontrivial cut of a symmetric directed network was given by Nagamochi and Ibaraki [61] (see also [80] and [33] for the simplified versions). In a theoretical point of view, it is worth considering whether the minimum cut algorithm of Nagamochi and Ibaraki can be generalized to the minimum cut problem on symmetric bidirected networks. Here, we call a bidirected network $\mathcal{N} = (G = (V, A), c)$ *symmetric* if for each arc $a \in A$ there exists an arc $a' \in A$ such that $\partial a' = -\partial a$ and $c(a) = c(a')$. In this case the reduction technique in Chapter 3 seems no longer to work. Generalizing the algorithm of Nagamochi and Ibaraki, Queyranne [70] showed a strongly polynomial combinatorial algorithm for minimizing a symmetric submodular function. A problem is naturally raised: Is there a purely combinatorial algorithm for minimizing a symmetric bisubmodular function in strongly polynomial time? Here, a function

$f: \mathcal{F} \to \mathbf{R}$ on a simple and spanning $\{\sqcup, \sqcap\}$-closed family $\mathcal{F} \subseteq 3^V$ is called *symmetric* if for each $(X, Y) \in \mathcal{F}$ we have $(Y, X) \in \mathcal{F}$ and $f(X, Y) = f(Y, X)$. The cut function associated with a symmetric bidirected network is symmetric and bisubmodular.

Concerning the problem of coloring the regions of the embedding of a graph into a surface, Bouchet [16] considered nowhere-zero circulations in bidirected graphs and showed that for any bidirected graph with no isthmus there is a nowhere-zero 126-circulation. The another result on nowhere-zero integral circulations was given by Khelladi [53]. These results are far from satisfactory compared with the results on ordinary directed graphs (see [49], [78]). We expect that the approach given in Chapter 3 is useful for obtaining a better bound.

In Chapter 4 we had seen the set theoretical version of the signed Birkhoff Theorem of V. Reiner, that is, there exists a one-to-one correspondence between the set of signed posets on a finite set V and the set of simple and spanning $\{\sqcup, \sqcap\}$-closed families on V, where each signed poset on V is made correspond to the set of its ideals. Based on our result, Iwata [48] gave a Dulmage-Mendelsohn decomposition of general (undirected) graphs. The decomposition is a refinement of Edmonds-Gallai decomposition of graphs and the components form signed posets. Signed poset is a so recently introduced concept that there are not many studies in them. We expect that our result will give us deeper insights for the other subjects related to signed posets and $\{\sqcup, \sqcap\}$-closed families as well.

In Chapter 5 we examined structures of bisubmodular polyhedra, such as the lineality spaces and the extreme rays, in terms of signed posets and exchangeability graphs. We considered linear optimization problems on possibly unbounded bisubmodular polyhedra. An optimality condition in terms of an exchangeability graph was shown and the greedy algorithm was examined. It was shown that a point in a bisubmodular polyhedron is an extreme point if and only if the associated exchangeability graph defines a signed poset. We designed an algorithm that discern whether a given arbitrary point is an extreme point or not. The algorithm also finds the signed poset associated with the point if it is an extreme point. Note that the complexity of the algorithm is $\mathrm{O}(|V|^2)$ and is the same as that for polymatroids ([15]), where we assume a function evaluation oracle for f. It follows that the adjacency of given two extreme points can be checked in the same complexity. We considered the decomposition of a bisubmodular system into its connected components and showed that each bisubmodular system decomposes into fully connected bisubmodular systems and connected submodular systems.

Unfortunately, the intersection of two bisubmodular polyhedra is not necessarily integral even if the associated bisubmodular functions are integer valued. Hence, if the intersection is not integral, it has no combinatorial significance and the study of the intersections seems to be less important. Nevertheless, it is known that the intersection is always half-integral and the characterization of the intersection of two bisubmodular polyhedra to be integral is a challenging problem. Partial results are shown in [38] and [52]. Also, generalizing concept of submodular flow one can consider a bisubmodular flow, i.e., a flow in a bidirected network such that its boundary is a point in a bisubmodular polyhedron. Flows in a "directed" network with their boundaries are elements of a "bisubmodular" polyhedron or flows in a "bidirected" network with their boundaries are in a "base polyhedron" are interesting special cases. An another special case was treated by Nakamura [63]. Little is known about bisubmodular flows but our investigation for the boundaries of flows in bidirected networks will make bisubmodular flows tractable.

In Chapter 6 we gave simple necessary and sufficient conditions for a function to be bisubmodular. Fujishige and Patkar [38] gave an another characterization of bisubmodular function: a function $f : 3^V \to \mathbf{R}$ is bisubmodular if and only if it is bisubmodular in each orthant and satisfies the *compliancy condition*. They considered polyhedra determined by functions on 3^V satisfying the compliancy condition and obtained an interesting result called the orthant non-interaction theorem. They also extended the notion of the Dilworth truncation of intersecting bisubmodular functions to the Dilworth truncation of compliant-intersecting bisubmodular functions, which is an interesting research topic.

In Chapter 7 we revealed the relationship between fractional degree-two polytopes and ideal polytopes of bidirected graphs. Actually, given a bidirected graph G, the fractional degree-two polytope and the ideal polytope associated with G are affine transformations of each other. This simple but definitely new observation provides us with a unifying framework for treating the other combinatorial structures such as quadratic Boolean equations and complete sets of implicants of lengths at most two, which are (almost) equivalent to degree-two constraints. From this isomorphism we have the half-integrality of fractional degree-two polytopes. The characterizations of integral fractional degree-two polytopes were described in terms of associated bidirected graph. Investigations of the face structures, adjacency, etc. remain to be done.

Valuated matroid ([24], [26]) is a generalization of matroid and the greedy algorithm for matroids is also generalized. The weighted matroid intersection problem has been extended by Murota ([56], [57]) to the valuated matroid intersection problem. The optimality criteria and the algorithm for the weighted matroid intersection problem are also generalized. This result has been reformulated in [58] into a novel min-max duality theorem. Recently, Murota ([59], [60]) reached the concept of combinatorial concave function. A combinatorial concave function is defined on the integral points of a base polyhedron and satisfies a certain exchange axiom, and it makes clear the relationship between the submodularity and the convexity generalizing the Lovász's observation [54]. We hope that the results obtained in this dissertation give a foundation for investigations of valuated delta-matroids ([25], [83]).

We mentioned in Chapter 1 that in recent years submodular functions have received considerable attentions in the theory of scheduling. In a one-machine scheduling problem without preemption, Queyranne [69] showed that the convex hull of the completion time vectors is an affine transformation of a supermodular polyhedron and that well-known Smith's algorithm is actually the greedy algorithm for the supermodular system. Although these scheduling problems are simple, we can use these structures as a basis for solving more complicated problems (see [82], [71]). In scheduling theory, there must exist scheduling problems where bisubmodular systems reveal the underlying structures.

The core of a convex game is a base polyhedron associated with its characteristic function ([79]). We expect roles played by bisubmodular functions and their associated polyhedra in game theory.

References

[1] R. K. Ahuja, T. L. Magnanti and J. B. Orlin: *Network Flows: Theory, Algorithms, and Applications* (Prentice Hall, New York, 1993).

[2] K. Ando: Some results on flows on bidirected graphs. Manuscript. April 1995.

[3] K. Ando: Fractional degree-two polytopes and ideal polytopes of bidirected graphs. Manuscript. August 1995.

[4] K. Ando and S. Fujishige: $\sqcup, \sqcap$-closed families and signed posets. Report No. 93813, Forschungsinstitut für Diskrete Mathematik, Universität Bonn (January 1994).

[5] K. Ando and S. Fujishige: On structures of bisubmodular polyhedra. Report No. 94821, Forschungsinstitut für Diskrete Mathematik, Universität Bonn (April 1994).

[6] K. Ando, S. Fujishige and T. Naitoh: A greedy algorithm for minimizing a separable convex function over an integral bisubmodular polyhedron. *Journal of the Operations Research Society of Japan* **37** (1994) 188-196.

[7] K. Ando, S. Fujishige and T. Naitoh: A characterization of bisubmodular functions. *Discrete Mathematics* (to appear).

[8] K. Ando, S. Fujishige and T. Naitoh: Proper bisubmodular systems and bidirected flows. Discussion Paper No. 532, Institute of Socio-Economic Planning, University of Tsukuba (April 1993).

[9] K. Ando, S. Fujishige and T. Naitoh: A greedy algorithm for minimizing a separable convex function over a finite jump system. *Journal of the Operations Research Society of Japan* **38** (1995) 362-375.

[10] K. Ando, S. Fujishige and T. Nemoto: Decomposition of a bidirected graph into strongly connected components and its signed poset structure. *Discrete Applied Mathematics* (to appear).

[11] K. Ando, S. Fujishige and T. Nemoto: The minimum-weight ideal problem for signed posets. *Journal of the Operations Research Society of Japan* (to appear).

[12] B. Aspvall, M. F. Plass and R. E. Tarjan: A linear-time algorithm for testing the truth of certain quantified Boolean formulas. *Information Processing Letters* **8** (1979) 121-123.

[13] N. Biggs: *Algebraic Graph Theory* (Cambridge University Press, London, 1974).

[14] G. Birkhoff: *Lattice Theory* (American Mathematical Colloquium Publications **25** (3rd ed.), Providence, R. I., 1967).

[15] R. E. Bixby, W. H. Cunningham and D. M. Topkis: The partial order of a polymatroid extreme point. *Mathematics of Operations Research* **10** (1985) 367-378.

[16] A. Bouchet: Nowhere-zero integral flows on a bidirected graph. *Journal of Combinatorial Theory*, Series B **34** (1983) 279-292.

[17] A. Bouchet: Greedy algorithm and symmetric matroids. *Mathematical Programming* **38** (1987) 147-159.

[18] A. Bouchet: Matchings and Δ-matroids. *Discrete Mathematics* **24** (1989) 55-62.

[19] A. Bouchet and W. H. Cunningham: Delta-matroids, jump systems and bisubmodular polyhedra. *SIAM Journal on Discrete Mathematics* **8** (1995) 17-32.

[20] A. Bouchet, A. Dress and T. Havel: Δ-matroids and metroids. *Advances in Mathematics* (to appear).

[21] R. Chandrasekaran and S. N. Kabadi: Pseudomatroids. *Discrete Mathematics* **71** (1988) 205-217.

[22] W. H. Cunningham and J. Green-Krótki: b-matching degree-sequence polyhedra. *Combinatorica* **11** (1991) 219-230.

[23] A. Dress and T. F. Havel: Some combinatorial properties of discriminants in metric vector spaces. *Advances in Mathematics* **62** (1986) 285-312.

[24] A. Dress and W. Wenzel: Valuated matroid: a new look at the greedy algorithm. *Applied Mathematics Letters* **3** (1990) 33-35.

[25] A. Dress and W. Wenzel: A greedy-algorithm characterization of valuated Δ-matroids. *Applied Mathematics Letters* **4** (1991) 55-58.

[26] A. Dress and W. Wenzel: Valuated matroids. *Advances in Mathematics* **93** (1992) 214-250.

[27] F. D. J. Dunstan and D. J. A. Welsh: A greedy algorithm solving a certain class of linear programmes. *Mathematical Programming* **62** (1973) 338-353.

[28] P. Edelman and R. Jamison: Theory of convex geometries. *Geometry Dedicata* **19** (1985) 247-270.

[29] J. Edmonds: Submodular functions, matroids, and certain polyhedra. In: *Combinatorial Structures and Their Applications* (R. Guy, H. Hanani, N. Sauer and J. Schönheim, eds., Gordon and Breach, New York, 1970), pp. 69-87.

[30] J. Edmonds and E. L. Johnson: Matching: a well-solved class of linear programs. In: *Combinatorial Structures and Their Applications* (R. Guy, H. Hanani, N. Sauer and J. Schönheim, eds., Gordon and Breach, New York, 1970), pp. 88-92.

[31] S. D. Fischer: *Signed Poset Homology and q-analog Möbius Functions.* Dissertation, University of Michigan, 1993.

[32] A. Frank: Generalized polymatroids. In: *Finite and Infinite Sets* (A. Hajnal, L. Lovász and V. T. Sós, eds., North-Holland, Amsterdam, 1984), pp. 285-294.

[33] A. Frank: On the edge-connectivity algorithm of Nagamochi and Ibaraki. Laboratoire Artemis, IMAG, Université J. Fourier, Grenoble, March 1994.

[34] S. Fujishige: The independent-flow problems and submodular functions (in Japanese). *Journal of the Faculty of Engineering, University of Tokyo* **A-16** (1978) 42-43.

[35] S. Fujishige: A note on Frank's generalized polymatroids. *Discrete Applied Mathematics* **7** (1984) 105-109.

[36] S. Fujishige: *Submodular Functions and Optimization* (North-Holland, Amsterdam, 1991).

[37] S. Fujishige: A min-max theorem for bisubmodular polyhedra. Report No. 94818, Forschungsinstitut für Diskrete Mathematik, Universität Bonn (February 1994).

[38] S. Fujishige and S. B. Patkar: The orthant non-interaction theorem for certain combinatorial polyhedra and its implications in the intersection and the Dilworth truncation of bisubmodular functions. *Optimization* **34** (1995) 329-339.

[39] D. R. Fulkerson: Networks, frames, blocking systems. In: *Mathematics of the Decision Sciences, Part I (Seminar Stanford, California, 1967)* (American Mathematical Society, Providence, 1967), pp. 303-334.

[40] R. Guha, S. N. Kabadi and P. Sharma: Bisubmodular polyhedra: Facets, adjacency and a greedy algorithm. Mathematical Programming Conference, Ann Arbor, Michigan, August 1994.

[41] P. L. Hammer and S. Nguyen: A partial order in the solution space of bivalent programs. In: *Combinatorial Optimization* (N. Christofides, A. Mingozzi, P. Toth and C. Sandi, eds., John Wiley & Sons, New York, 1979), pp. 93-106.

[42] P. Hansen and B. Jaumard: Uniquely solvable quadratic Boolean equations. *Discrete Applied Mathematics* **12** (1985) 147-154.

[43] F. Harary: On the notion of balance of a signed graph. *Michigan Mathematical Journal* **2** (1953-1954) 143-146. Addendum, same journal preceding p. 1.

[44] D. Hausmann: *Adjacency on Polytopes in Combinatorial Optimizations* (Oelgeschlager, Gunn & Hain, Cambridge, 1979).

[45] D. Hausmann and B. Korte: Colouring criteria for adjacency on 0-1 polyhedra. *Mathematical Programming Study* **8** (1978) 106-127.

[46] Y. T. Ikebe and A. Tamura: Ideal polytopes and face structures of some combinatorial optimization problems. *Mathematical Programming* **71** (1995) 1-16.

[47] M. Iri and S. Fujishige: Use of matroid theory in operations research, circuits and system theory. *International Journal of Systems Science* **12** (1981) 27-54.

[48] S. Iwata: Dulmage-Mendelsohn type decomposition for general graphs. Working Paper RIMS-1017, Research Institute for Mathematical Sciences, Kyoto University.

[49] F. Jaeger: Flows and generalized coloring theorem in graphs. *Journal of Combinatorial Theory*, Series B **26** (1979) 205-216.

[50] E. L. Johnson and M. W. Padberg: Degree-two inequalities, clique facets, and biperfect graphs. *Annals of Discrete Mathematics* **16** (1982) 169-187.

[51] S. N. Kabadi: *Characterization and Development of Solution Methods for Special Classes of Totally Dual Integral Systems.* Dissertation, School of Management, University of Texas at Dallas, 1984.

[52] S. N. Kabadi and R. Chandrasekaran: On totally dual integral systems. *Discrete Applied Mathematics* **26** (1990) 87-104.

[53] A. Khelladi: Nowhere-zero integral chains and flows in bidirected graphs. *Journal of Combinatorial Theory*, Series B **43** (1987) 95-115.

[54] L. Lovász: Submodular functions and convexity. In: *Mathematical Programming – The State of the Art* (A. Bachem, M. Grötschel and B. Korte, eds., Springer, Berlin, 1983), pp. 235-257.

[55] K. Murota: *System Analysis by Graphs and Matroids — Structural Solvability and Controllability* (Algorithms and Combinatorics **3**) (Springer, Berlin, 1987).

[56] K. Murota: Valuated matroid intersection, I: Optimality criteria. Report No. 95837, Forschungsinstitut für Diskrete Mathematik, Universität Bonn (January 1995).

[57] K. Murota: Valuated matroid intersection, II: Algorithms. Report No. 95838, Forschungsinstitut für Diskrete Mathematik, Universität Bonn (January 1995).

[58] K. Murota: Fenchel-type duality for matroid valuations. Report No. 95839, Forschungsinstitut für Diskrete Mathematik, Universität Bonn (January 1995).

[59] K. Murota: Submodular flow problem with a nonseparable cost function. Report No. 95843, Forschungsinstitut für Diskrete Mathematik, Universität Bonn (March 1995).

[60] K. Murota: Convexity and Steinitz's exchange property. Report No. 95848, Forschungsinstitut für Diskrete Mathematik, Universität Bonn (June 1995).

[61] H. Nagamochi and T. Ibaraki: Computing edge-connectivity in multigraphs and capacitated graphs. *SIAM Journal on Discrete Mathematics* **5** (1992) 54-66.

[62] M. Nakamura: A characterization of greedy sets: Universal polymatroids (I). Scientific Papers of the College of Arts and Sciences, University of Tokyo, **38**-2 (1988) pp. 155-167.

[63] M. Nakamura: Δ-polymatroids and an extension of Edmonds-Giles' TDI scheme. *Proceedings of the Third IPCO Conference* (1993), pp. 401-412.

[64] G. L. Nemhauser and L. E. Trotter, Jr.: Vertex packings: structural properties and algorithms. *Mathematical Programming* **8** (1975) 232-248.

[65] J. C. Picard: Maximal closure of a graph and applications to combinatorial problems. *Management Science* **22** (1976) 1268-1272.

[66] J. C. Picard and M. Queyranne: Selected applications on minimum cuts in networks. *INFOR* **20** (1982) 394-422.

[67] L. Qi: Directed submodularity, ditroids and directed submodular flows. *Mathematical Programming* **42** (1988) 579-599.

[68] L. Qi: Bisubmodular functions. CORE Discussion Paper No. 8901, CORE, Université Catholique de Louvain, 1989.

[69] M. Queyranne: Structure of a simple scheduling polyhedra. *Mathematical Programming* **58** (1993) 263-285.

[70] M. Queyranne: A combinatorial algorithm for minimizing a symmetric submodular function. *Proceedings of the Sixth Annual ACM-SIAM Symposium on Discrete Algorithms* (1995), pp. 98-101.

[71] M. Queyranne and A. S. Schulz: Polyhedral approaches to machine scheduling. No. 408/1994, Technische Universität Berlin (November 1994).

[72] A. Recski: *Matroid Theory and its Applications in Electric Networks Theory and Statistics* (Algorithms and Combinatorics **6**) (Springer, Berlin, 1989).

[73] V. Reiner: Signed posets. *Journal of Combinatorial Theory*, Series A **62** (1993) 324-360.

[74] R. T. Rockafellar: The elementary vectors of a subspace of R^N. In: *Combinatorial Mathematics and Its Applications* (R. C. Bose and T. A. Dowling, eds., The University of North Carolina Press, Chapel Hill, 1969), pp. 104-127.

[75] S. Rudeanu: Boolean functions and generalized minterms. *Discrete Applied Mathematics* **49** (1983) 241-248.

[76] A. Schrijver: Matroids and linking systems. *Journal of Combinatorial Theory*, Series B **26** (1979) 349-369.

[77] A. Schrijver: *Theory of Linear and Integer Programming* (John Wiley & Sons, New York, 1986).

[78] P. D. Seymour: Nowhere-zero 6 flows. *Journal of Combinatorial Theory*, Series B **30** (1981), 130-135.

[79] L. S. Shapley: Cores of convex games. *International Journal of Game Theory* **1** (1971) 11-26.

[80] M. Stoer and F. Wagner: A simple min-cut algorithm. *Proceedings of the 1994 European Symposium on Algorithms ESA'94* (1995), pp. 141-147.

[81] J. Stoer and C. Witzgall: *Convexity and Optimization in Finite Dimensions I* (Springer, Berlin, 1970).

[82] A. Wang: A polyhedral approach to job sequencing. Working Paper (January 1995).

[83] W. Wenzel: Pfaffian forms and Δ-matroids. *Discrete Mathematics* **115** (1993) 253-266.

[84] H. Whitney: On the abstract properties of linear dependence. *American Journal of Mathematics* **57** (1935) 509-533.

[85] T. Zaslavsky: Signed graphs. *Discrete Applied Mathematics* **4** (1982) 47-74. Erratum, *ibid.*, **5** (1983) 248.

[86] T. Zaslavsky: Orientation of signed graphs. *European Journal of Combinatorics* **12** (1991) 361-375.