

The Greedy Algorithm for a Class of Lattice Polyhedra and its Consequences

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ISMP 2000

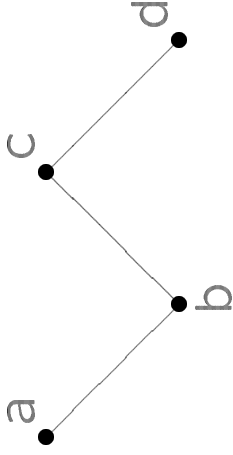
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Outline of this Talk

1. Review of submodular system
2. Review of submodular functions on antichains and Dual Greedy Algorithm
3. Lovász extension
4. An algorithm for testing a vector for being an extreme point
5. Concluding Remarks

Submodular Systems

- $P = (E, \preceq)$: partially ordered set (poset). ($n = |E|$.)



- $I \subseteq E$ is an **ideal** of $P \Leftrightarrow (y \preceq x \in I \Rightarrow y \in I)$.
 $\mathcal{I}(P) =$ the set of ideals of P .

Remark: $\mathcal{I}(P)$ forms a distributive lattice with lattice operations \cup and \cap .

- $f: \mathcal{I}(P) \rightarrow \mathbf{R}$ is **submodular** \Leftrightarrow
 $\forall A, B \in \mathcal{I}(P): f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$.

- Validity of Greedy Algorithm for solving

$$\left| \begin{array}{l} \max \sum \{w(e)x(e) \mid e \in E\} \\ \text{s.t. } x(X) \leq f(X) \quad (X \in \mathcal{I}(P)), \end{array} \right.$$

where $x(A) = \sum_{e \in A} x(e)$.

- $f: \mathcal{I}(P) \rightarrow \mathbf{R}$ is submodular iff its Lovász extension is convex.
- Existence of polynomial time algorithm for minimizing f .
- TDIity of $x(X) \leq f(X)$ ($X \in \mathcal{I}(P)$).
- Intersection Theorem
- ...

Submodular Functions on Antichains

- $A \subseteq E$ is an *antichain* of $P = (E, \preceq)$ if $\forall a, b \in A: a \preceq b \Rightarrow a = b$.
- $\mathcal{A}(P)$ the set of antichains of P .
- For $f: \mathcal{A}(P) \rightarrow \mathbf{R}$ define

$$P(f) = \{x \mid x \in \mathbf{R}^E, \forall A \in \mathcal{A}(P): x(A) \leq f(A)\}.$$

Validity of Dual Greedy Algorithm

$$(P) \quad \left| \begin{array}{l} \max \sum \{w(e)x(e) \mid e \in E\} \\ \text{s.t. } x(A) \leq f(A) \quad (A \in \mathcal{A}(P)) \end{array} \right.$$

- P : rooted forest + $f: \mathcal{A}(P) \rightarrow \mathbf{R}$: submodular [Faigle and Kern (1996)]
- $f: \mathcal{A}(P) \rightarrow \mathbf{R}$: submodular and monotone [Faigle and Kern (2000)]
- $f: \mathcal{A}(P) \rightarrow \mathbf{R}$: b -submodular [Krüger (2000)]

Ideals and Antichains

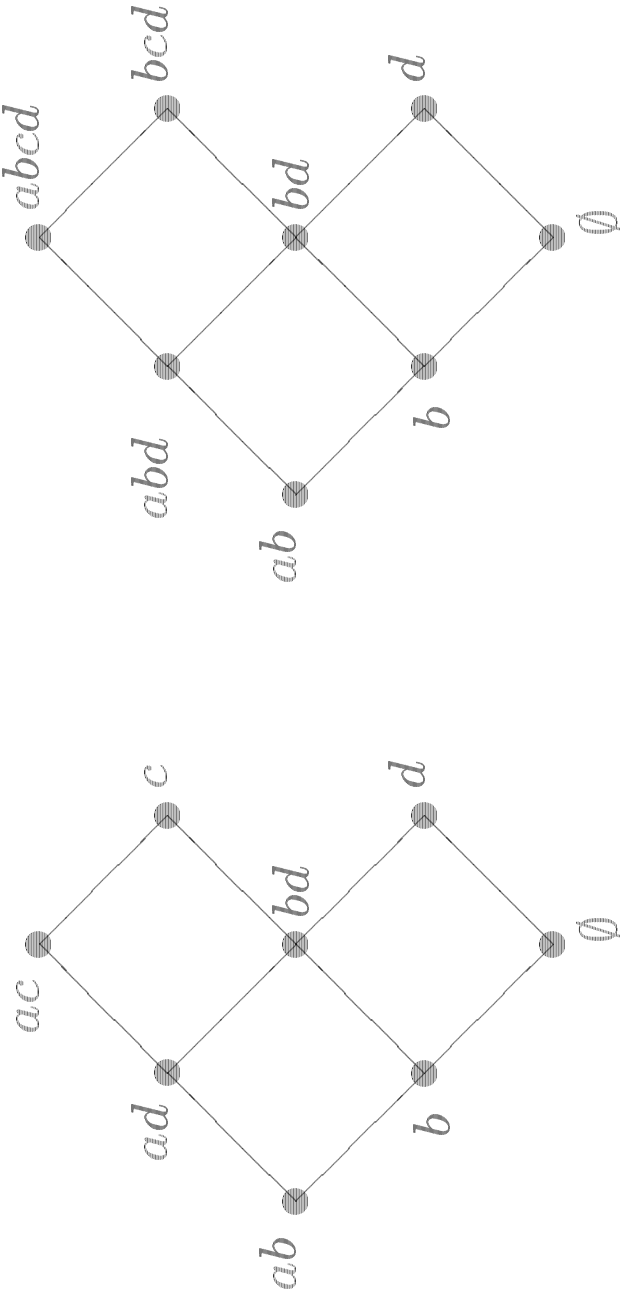
- The ideal generated by $A \in \mathcal{A}(P)$

$$\text{id}(A) = \{y \mid y \in E, \exists x \in A: y \preceq x\} \in \mathcal{I}(P).$$

- For $I \in \mathcal{I}(P)$,

$$\max(I) = \{x \mid x \in I, \nexists y \in I: x \prec y\} \in \mathcal{A}(P).$$

Proposition: The mappings $\text{id}: \mathcal{A}(P) \rightarrow \mathcal{I}(P)$ and $\max: \mathcal{I}(P) \rightarrow \mathcal{A}(P)$ are one-to-one correspondences. Each of them is the inverse of the other. \square



$$A(P) \xrightleftharpoons[\max]{\text{id}} \mathcal{I}(P)$$

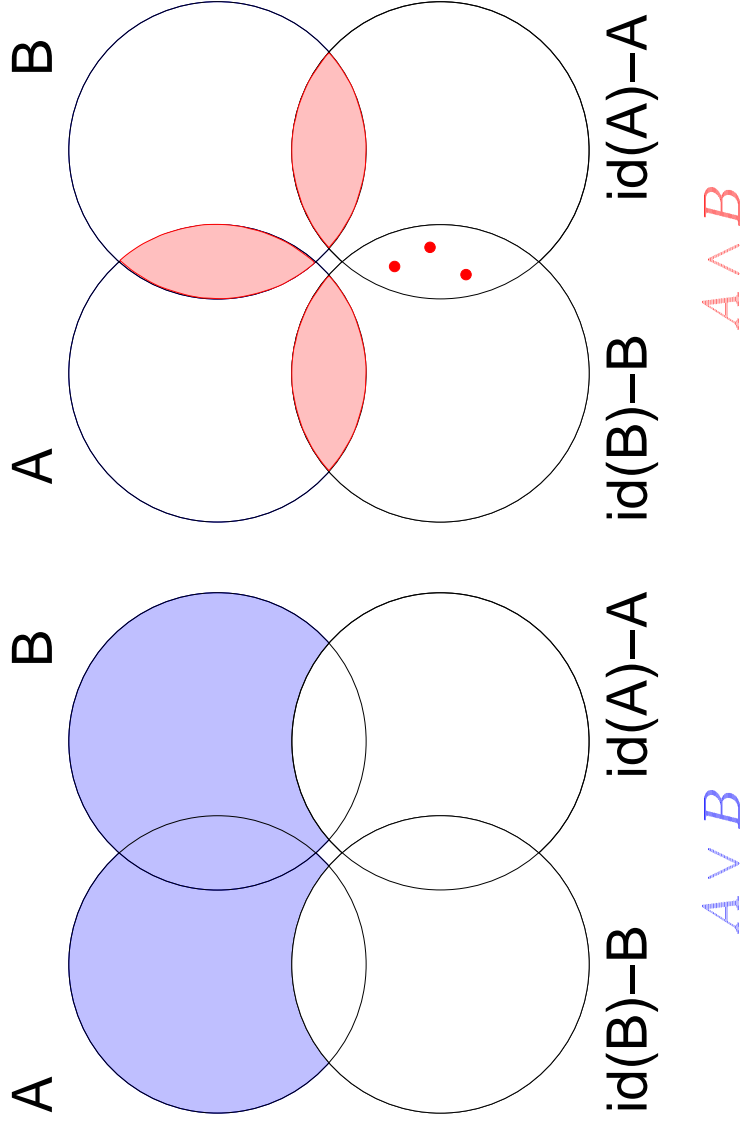
Corollary: The bijections induce lattice operations on $\mathcal{A}(P)$: $(\mathcal{I}(P); \cup, \cap) \cong (\mathcal{A}(P); \vee, \wedge)$, where

$$A \vee B = \max(\text{id}(A) \cup \text{id}(B)),$$

$$A \wedge B = \max(\text{id}(A) \cap \text{id}(B)).$$

and partial order \preceq on $\mathcal{A}(P)$ as

$$\forall A, B \in \mathcal{A}(P): A \preceq B \Leftrightarrow \text{id}(A) \subseteq \text{id}(B). \quad \square$$



Dual Greedy Algorithm

- $f: \mathcal{A}(P) \rightarrow \mathbf{R}$ is *submodular* $\Leftrightarrow A, B \in \mathcal{A}(P)$:
$$f(A) + f(B) \geq f(A \vee B) + f(A \wedge B).$$
- The polyhedron associated with $f: \mathcal{A}(P) \rightarrow \mathbf{R}$
$$P(f) = \{x \mid x \in \mathbf{R}^E, \forall A \in \mathcal{A}(P): x(A) \leq f(A)\}.$$
- LP over $P(f)$ and its dual

$$\begin{array}{l|l} \text{(P)} & \begin{array}{l} \max \sum \{w(e)x(e) \mid e \in E\} \\ \text{s.t. } x(A) \leq f(A) \quad (A \in \mathcal{A}(P)) \end{array} \\ \text{(D)} & \begin{array}{l} \min \sum \{f(A)y(A) \mid A \in \mathcal{A}(P)\} \\ \text{s.t. } \sum \{y(A) \mid e \in A \in \mathcal{A}(P)\} = w(e) \quad (e \in E), \\ y(A) \geq 0 \quad (A \in \mathcal{A}(P)) \end{array} \end{array}$$

$$\begin{pmatrix} A_1 & e_1 & e_2 & e_3 & \cdots & e_n \\ 1 & 0 & \cdots & \cdots & \cdots & 0 \\ * & 1 & 0 & \cdots & \cdots & 0 \\ * & * & 1 & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \cdots & 0 \\ A_n & * & * & \cdots & \cdots & 1 \end{pmatrix} = \begin{bmatrix} x(e_1) \\ x(e_2) \\ \vdots \\ \vdots \\ x(e_n) \end{bmatrix} \begin{bmatrix} f(A_1) \\ f(A_2) \\ \vdots \\ \vdots \\ f(A_n) \end{bmatrix}$$

where each row is the characteristic vector χ_{A_i} of A_i .

For $X \subseteq E$ define the characteristic vector $\chi_X: E \rightarrow \{0, 1\}$ by

$$\chi_X(e) = \begin{cases} 1 & \text{if } e \in X, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma: y^* is feasible for (D) . \square

Algorithm 1 Dual Greedy (Faigle and Kern(1996))

- 1: $y^* \leftarrow 0$.
- 2: **for** $i = n$ **downto** 1 **do**
- 3: Choose $e_i \in \max(E)$ such that
 $w(e_i) = \min\{w(e) \mid e \in \max(E)\}$.
- 4: $A_i \leftarrow \max E$.
- 5: $y^*(A_i) \leftarrow w(e_i)$.
- 6: $w(e) \leftarrow w(e) - w(e_i)$ ($e \in \max(E)$).
- 7: $\{w(e)$ is kept nonnegative and so is $y^*.\}$
- 8: $E \leftarrow E - \{e_i\}$.
- 9: **end for**
- 10: Define x^* as the unique solution of the system
$$x(A_i) = f(A_i) \quad (i = 1, \dots, n)$$

of equations.
- 11: $\{x^*$ and y^* satisfies complementary slackness.

Theorem (Faigle and Kern 1996): If $P = (E, \preceq)$ is a rooted forest and $f: \mathcal{A}(P) \rightarrow \mathbf{R}_+$, then $x^* \in \mathcal{P}(f)$ is feasible for (P) , and hence, Dual Greedy Algorithm is valid for any $w: E \rightarrow \mathbf{R}_+$.

Theorem (Faigle and Kern 2000): Let $P = (E, \preceq)$ be an arbitrary poset and f is submodular and monotone, then Dual Greedy Algorithm is valid for any $w: E \rightarrow \mathbf{R}_+$.

Submodularity in Krüger's sense

- $f: A(P) \rightarrow \mathbb{R}$ **b-submodular**
 $\Leftrightarrow \forall A, B \in A(P)$:

$$f(A) + f(B) \geq f(A \vee B) + f(A \wedge B),$$

where

$$A \wedge B = (A \cap B) \cap (A \cup B).$$

Remark: If P is a rooted forest, we have $A \cap B = A \wedge B$, and hence, submodularity and b-submodularity are equivalent.

Theorem (Krüger 2000): For any poset P , if $f: A(P) \rightarrow \mathbb{R}$ b-submodular, Dual Greedy Algorithm is valid for each $w: E \rightarrow \mathbb{R}_+$. \square

Corollary: Let P be an arbitrary poset. For a function $f: A(P) \rightarrow \mathbb{R}$ Dual Greedy Algorithm is valid for each $w: E \rightarrow \mathbb{R}_+$ if and only if f is b-submodular. \square

The Lovász extension

Lemma: For each $w: E \rightarrow \mathbf{R}_+$, there uniquely exists a chain

$$C: A_1 \prec A_2 \prec \cdots \prec A_k$$

of nonempty antichains of P and $\lambda_i > 0$ ($i = 1, \dots, k$) such that

$$w = \lambda_1 \chi_{A_1} + \lambda_2 \chi_{A_2} + \cdots + \lambda_k \chi_{A_k}, \quad (1)$$

where $k \geq 0$. \square

- Lovász extension

Let $P = (E, \preceq)$ be a poset and consider an arbitrary function $f: A(P) \rightarrow \mathbf{R}$. Define

$$\hat{f}(w) = \sum_{i=1}^k \lambda_i f(A_i), \quad (2)$$

where w has the uniquely represented as (1).

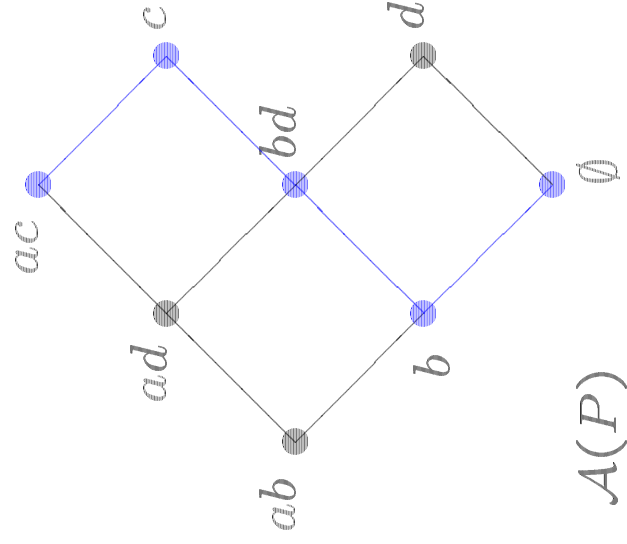
Theorem: The Lovász extension $\hat{f}: \mathbf{R}_+^E \rightarrow \mathbf{R}$ is convex if and only if f is b -submodular. \square

Algorithm for testing a vector for being an extreme point

Theorem (Krüger 2000): $x \in \mathbf{R}^E$ is an extreme point of $P(f)$ if and only if there exists a chain

$$C: \emptyset = A_0 \prec A_1 \prec \dots \prec A_{n-1} \prec A_n = \max(E)$$

of antichains such that $x(A_i) = f(A_i)$ for $i = 1, \dots, n$. \square



- $A \leftarrow \emptyset$.
- while $\exists B \in A(P)$ s.t. B covers A
and $x(B) = f(B)$ do $A \leftarrow B$.

Lemma: Let $A \in \mathcal{A}(P) - \{\max(E)\}$. Then, $A \prec \cdot B$ if and only if $B = A \vee e$ for some $e \in \min(E - \text{id}(A))$. \square

Lemma: Suppose that x is an extreme point of $P(f)$. For each $A \in \mathcal{A}(P) - \{\max(E)\}$ with $x(A) = f(A)$ there exists an $e \in \min(E - \text{id}(A))$ such that $x(A \vee e) = f(A \vee e)$. \square

Theorem: The following algorithm is valid. \square

Algorithm 2 Extreme

Require: $x \in \mathbb{R}^E$.

Ensure: YES if x is an extreme point of $P(f)$, NO otherwise.

- 1: $A \leftarrow \emptyset$.
 - 2: **while** $\exists e \in \min(E - \text{id}(A))$ such that $x(A \vee e) = f(A \vee e)$ **do**
 - 3: $A \leftarrow A \vee e$.
 - 4: **end while**
 - 5: **if** $A = \max(E)$ **then**
 - 6: return YES.
 - 7: **else**
 - 8: return NO.
 - 9: **end if**
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Conclusion and Remarks

- Characterization by the validity of Dual Greedy Algorithm.
- TDIity.
- Lovász extension.
- Polynomial time algorithm for b-submodular function minimization.
- Combinatorial Polynomial time algorithm for testing a vector for being an extreme point.

- Intersection Theorem \implies holds if P is a rooted forest (Faigle and Kern (2000)).
- Intersection of $P(f)$ and a box.
- Representation of $P(f)$ by a submodular flow polyhedron [Fujishige (2000)].