劣モジュラ費用の配分メカニズム

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Outline

- Cost games
- Cost sharing methods
- Cost sharing mechanisms
- Concluding remarks

References

• A. Claus and D. J. Kleitman: Cost allocation for a spanning tree. *Networks* **3** (1973) 289--304.
• B. Dutta: The egalitarian solution and reduced game properties in convex games. *Internat. J. Game Theory* **19** (1990) 153--169.
• B. Dutta and D. Ray: A concept of egalitarianism under participation constraints. *Econometrica* **57** (1989) 615-635.
• J. Feigenbaum, C. H. Papadimitriou and S. Shenker: Sharing the cost of multicast transmissions. *J. Comput. System Sci.* 63 (2001) 21-41.
• D. Granot and G. Huberman: The relationship between convex games and minimum cost spanning tree games: A case for permutationally convex games. *SIAM J. Alg. Disc. Meth*. **3** (1982) 288-292.

References

• J. Green and J. J. Laffont: Incentives in public decision making. In: *Studies in Public Economics*, Vol. 1 (North-Holland, 1979).
• H. Moulin and S. Shenker: Strategyproof sharing of submodular costs: budget balance versus efficiency. *Economic Theory* **18** (2001) 511-533.
• S. Mutuswami: Strategyproof mechanisms for cost sharing. Working paper, Indial Statistical Institute, 1997.
• H. Norde, S. Moretti and S. Tijs: Minimum cost spanning tree games and population monotonic allocation schemes. *European J. Oper. Res.* **154** (2004) 84--97.
• Y. Sprumont: Population monotonic allocation schemes for cooperative games with transferable utility. *Games Econom. Behav*. **2** (1990) 378--394.

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Cost games

•An *n*-person cooperative game is a pair (*N*,*C*) of set *N*={1,2,…,*n*} of players (or agents, users) and a function $C:2^N \to R$ with

$C(\phi)=0$ and

$C(S) \le C(T)$ for $S \subseteq T \subseteq N$.

•For $S \subseteq N$ a game (S, C_S) is called subgame induced by *S*, where $C_S : 2^S \to R$ is the restriction of *C* to 2^S :

$$C_S(T) = C(T) \quad (T \subseteq S).$$

Core

•The core $\text{core}(C)$ of a game (N,C) is defined by

$$\text{core}(C) = \{x \in R^N \mid \forall T \subseteq N : x(T) \le C(T),\; x(N) = C(N)\},$$

where $x(T) = \sum_{i \in T} x_i$ for each $T \subseteq N$.

Submodular functions

•A function $C : 2^N \to R$ is submodular if for each $S,T \subseteq N$

$$C(S) + C(T) \ge C(S \cup T) + C(S \cap T).$$

•A game (N, C) is concave if C is submodular.

Example 1: MCST games* [Claus-Kleitman '73]

•Let $N' = N \cup \{0\}$ and $G=(N',E)$ be the complete graph with length function $l : E \to R_+$.

•Define $C : 2^N \to R$ by

$$C(S) = \min\{c(T) \mid T \text{ is a spanning tree of } G[S \cup \{0\}]\},$$

where $G[S \cup \{0\}]$ is the subgraph of G induced by $S \cup \{0\} \subseteq N'$.

Remark: The cost function C is not necessarily submodular but is permutationally submodular. [Granot-Huberman '82]

Permutational submodularity

•For a permutation π of N define

$$S_j^\pi = \{i_1, i_2, \cdots, i_j\} \quad (j = 1, \cdots, n),$$

where

$$\pi(i_1) < \pi(i_2) < \cdots < \pi(i_n).$$

•A function $C : 2^N \to R$ is permutationally submodular if there exists a permutation π of N such that for each $j < k$ *and* $T \subseteq N - S_k^\pi$

$$C(S_j^\pi \cup T) + C(S_k^\pi) \ge C(S_k^\pi \cup T) + C(S_j^\pi).$$

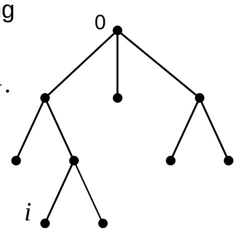
Remark: If C is permutationally submodular, then (N,C) has a nonempty core. [Granot-Huberman '82]

Example 2: Fixed tree multicast routing [Feigenbaum et al. '01]

•Let $T = (V, E)$ be a tree rooted at 0 with length function being $l : E \to R_+$.

•We assume $N \subseteq V - \{0\}$.

•For each $i \in N$ let $T(i)$ be the set of edges in the unique path from i to 0.

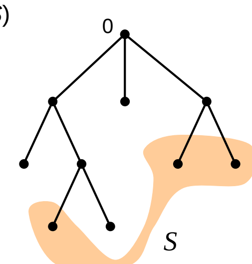


Example 2: Fixed tree multicast routing (cnt'd)

•For each $S \subseteq N$ define $C(S)$ by

$$C(S) = l(\cup_{i \in S} T(i)).$$

Remark: The function $C : 2^N \to R$ is submodular.



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Cost sharing method

Given a cost game (N, C) a mapping

$$\xi : 2^N \times N \to R_+$$

is called a cost sharing method for (N, C) if for each $S \subseteq N$

$$\sum_{i \in S} \xi_i(S) = C(S),$$
$$\xi_i(S) = 0 \quad \forall i \notin S.$$

Cross-monotonicity

A cost sharing method **ξ** is called cross-monotonic if

$$i \in S \subseteq T \Rightarrow \xi_i(S) \geq \xi_i(T).$$

Remark: If **ξ** is a cross-monotonic csm, then **ξ** (S)∈Core(C_S) and so (N,C) is totally balanced, i.e., the core of each subgame of (N,C) is nonempty.The converse if not true when n≧4. [Sprumont '90]

Marginal contribution vector

•Let **π** be a permutation of N. For S⊆N let

$$S = \{i_1, \cdots, i_k\},$$

where $\pi(i_1) < \cdots < \pi(i_k)$.

•Define $\xi^\pi(S)$ by

$$\xi^\pi_{i_j}(S) = C(\{i_1, \cdots, i_j\}) - C(\{i_1, \cdots, i_{j-1}\})$$

for $j = 1, \cdots, k$.

The csm ξ^π is cross-monotonic if C is submodular. [cf. Sprumont '90]

Shapley value

The csm ξ^* defined by

$$\xi^*(S) = \frac{1}{n!}\sum_{\pi} \xi^\pi(S) \qquad (S \subseteq N)$$

is called the Shapley value, where the sum is over all the permutations of N.

The Shapley value ξ^* is cross-monotonic if C is submodular. [cf. Sprumont '90]

Fujishige-Dutta-Ray solution (Egalitarian solution)

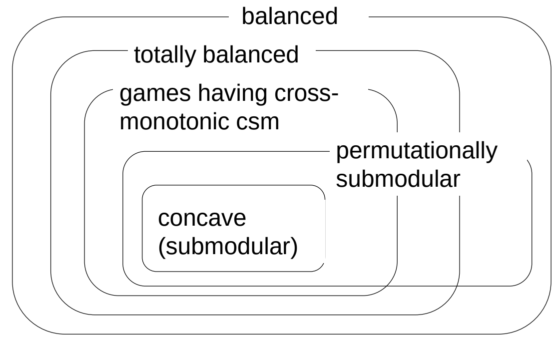
If C is submodular, the Fujishige-Dutta-Ray solution is cross-monotonic. [Dutta '90]

Cross-monotonic csm for MCST games

Bird rule is NOT a cross-monotonic csm but

Theorem [Norde et al. '04]: Every MCST game has a cross-monotonic csm.

Summary of this section



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Problem Setting

- A service provider is about to provide a service to a subset of

  N: the set of agents (or players, users).
- The cost for the provider to give the service to each $S \subseteq N$ is given by $C : 2^N \to R$.
- Each agent $i \in N$ report $u_i \geq 0$, her willingness to pay, which may not be true, for the service.
- The provider must decide which agent receive the service, and how much they are charged.

Cost sharing mechanism

A cost sharing mechanism is a mapping

$$M : R_+^N \to 2^N \times R^N$$

associating to each profile $u \in R_+^N$ of willingness to pay, a pair $(Q(u), x(u))$, where

$Q(u) \in 2^N$ (the set of agents who are served),

$x(u) \in R^N$ (the charge each agent must pay).

Individual welfare*

Agent *i*'s individual welfare $w_i(u)$ is defined by

$$w_i(u) = q_i(u)u_i - x_i(u),$$

where $q(u) \in R^N$ is defined as

$$q_i(u) = \begin{cases} 1 & \text{if } i \in Q(u), \\ 0 & \text{otherwise.} \end{cases}$$

Strategy-proofness

- For $u \in R_+^N, i \in N$ and $u' \in R_+$ define

$$(u', u_{-i})_j = \begin{cases} u' & \text{if } j = i, \\ u_j & \text{otherwise.} \end{cases}$$

- Strategy-proofness: A mechanism *M* is strategy-proof if

$$q_i(u)u_i - x_i(u) \geq q_i(u', u_{-i})u_i - x_i(u', u_{-i})$$

for all $u \in R_+^N, i \in N$, and $u' \in R_+$.

Group strategy-proofness

- For $u \in R_+^N, S \subseteq N$ and $u' \in R_+^S$ define

$$(u', u_{-S})_j = \begin{cases} u'_j & \text{if } j \in S, \\ u_j & \text{otherwise.} \end{cases}$$

•Group strategy-proofness: A mechanish *M* is group strategy-proof if
for all $u \in R_+^N, S \subseteq N$, and $u' \in R_+^S$,

$$\forall i \in S: q_i(u)u_i - x_i(u) \leq q_i(u', u_{-S})u_i - x_i(u', u_{-S})$$

implies equality in the above for each *i*∈*S*.

Requirements of a mechanism*

•(Group) strategy-proofness
•No Positive Transfer (NPT):

$$x_i(u) \geq 0 \quad \text{for all } u \text{ and } i.$$

• Voluntary Participation (VP): for all *i* and for all $u \in R_+^N$

$$q_i(u)u_i - x_i(u) \geq 0.$$

•Customer Sovereignty (CS):

$$\forall i \in N, \exists u_i \in R_+ \text{ such that } i \in Q(u_i, u_{-i}) \text{ for all } u_{-i}.$$

Requirements of a mechanism (cnt'd)

•Budget-balance:

$$\sum_{i \in S(u)} x_i(u) = C(S(u)) \text{ for all } u \in R_+^N.$$

• Efficiency:

$$u(S(u)) - C(S(u)) = \max\{ u(T) - C(T) \mid T \subseteq N \}$$

Remark: There is no strategy-proof mechanism that is both budget-balanced and efficient. [Green-Laffont '79]

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Mechanism *M*(ξ)

Let $\xi: 2^N \times N \to R_+$ be a cross-monotonic cost sharing method, the cost sharing mechanism $M(\xi)$ is defined by the following procedure.

1. $Q := N$.
2. While there exists $i \in Q$ such that $u_i < \xi_i(Q)$ delete i from Q.
3. Set $Q(u) := Q$ and

$$x_i(u) = \begin{cases} \xi_i(Q) & \text{if } i \in Q, \\ 0 & \text{otherwise.} \end{cases}$$

Moulin-Shenker Theorem

Theorem [Moulin et al. '01]:
(i) For any cross-monotonic cost sharing method ξ, the mechanism $M(\xi)$ is budged-balanced, meets NTP, VP and CS and is group strategy-proof.
(ii) Conversely, for any mechanism M satisfying budged-balance, NTP, VP, CS and group strategy-proofness, there exists a cross-monotonic cost sharing method ξ such that $M(\xi)$ is welfare equivalent to M.

Welfare equivalence

Two mechanism M and M' are welfare equivalent if for all $u \in R_+^N$ and $i \in N$

$$q_i(u)u_i - x_i(u) = q'_i(u)u_i - x'_i(u),$$

where

$$M : u \mapsto (q(u), x(u)) \text{ and}$$
$$M' : u \mapsto (q'(u), x'(u)).$$

Maximum efficiency loss

•Let

$$w(N,u) = \max\{ u(T) - C(T) \mid T \subseteq N \}$$

and define

$$\gamma(\xi,u) = w(N,u) - [u(\tilde{Q}(\xi,u)) - C(\tilde{Q}(\xi,u))].$$

•The maximum efficiency loss $\gamma(\xi)$ of the mechanism ξ is defined as

$$\gamma(\xi) = \sup\{ \gamma(\xi,u) \mid u \in R_+^N \}.$$

Maximum efficiency loss

Theorem [Moulin et al. '01]: The Shapley value mechanism $M(\xi^*)$ is the unique minimizer of γ among all the mechanisms derived from cross monotonic cost sharing methods.

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Efficient sets

• A subset $S \subseteq N$ is called efficient if

$$u(S) - C(S) = \max\{ u(T) - C(T) \mid T \subseteq N \}.$$

• The efficient sets are closed under ∪ and ∩, and so, there exists the largest efficient set $Q^*(u)$.

•Define

$$w(N,u) = \max\{ u(T) - C(T) \mid T \subseteq N \}.$$

MC mechanism

•Define

$$w(N-i,u) = \max\{\, u(T) - C(T) \mid T \subseteq N-i \}.$$

•The marginal cost pricing mechanism (MC mechanism) picks the coalition $Q^*(u)$ and cost share $x^*(u)$ defined as

$$x_i^*(u) = u_i q_i^*(u) - (w(N,u) - w(N-i,u)).$$

MC mechanism

Theorem [Moulin et al. '01]:
(i) The MC mechanism meets NPT, VP and CS and is strategyproof.
(ii) Conversely, any strategy-proof and efficient mechanism meeting NPT, VP and CS is welfare equivalent to MC.

Remark: The MC mechanism is not group strategy-proof.

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•Does Fujishige-Dutta-Ray mechanism have a nice characterization as Shapley value mechanism? (Partially answered by [Mutuswami '97].)
•Algorithms to implement a mechanism for combinatorial games. (Algorithm implementing MC mechanism [Feigenbaum et al. '01] is given for multicast routing.)